

A SCHOOL
ALGEBRA

PART III

H. S. HALL



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EXAMPLES IN ALGEBRA

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A SCHOOL ALGEBRA

PART III

BY

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PREFATORY NOTE TO PART III.

THIS concluding volume begins with Permutations and Combinations and carries the subject as far as is deemed necessary for an ordinary School Course.

The complete work contains all the essentials of Elementary Algebra, and will be found amply sufficient for all students who are not specializing in mathematics. Those who are destined to become mathematicians in any real sense will pursue the study of Algebra in more advanced works, and in due course will fill some gaps which form part of a deliberate plan in the present text-book. I refer, in particular, to the latter half of Chap. XLI., pages 482-494, and Chap. XLIII., where I have emphasized the *practical use* of the Binomial, Exponential, and Logarithmic Series, though omitting proofs of theorems which cannot be established satisfactorily without a considerable digression on Convergency and Divergency of Series.

The difficulties connected with the Exponential Theorem and Logarithmic Series were discussed a few years ago at a meeting of the British Association, and in some subsequent papers in the *Mathematical Gazette*. Among the methods of proof there suggested there was not one which was in the least suitable for an elementary book. Further, from the views expressed by some eminent mathematicians who took part in the discussion, I quote the following remarks :

“The less beginners are troubled with questions of convergency of series the better.”

“To base the exponential theorem on the limit of $\left(1 + \frac{x}{n}\right)^n$, when n is infinite, is logically quite wrong. The logarithmic expansion is more difficult and may well be postponed for a while.”

After a personal experience of nearly thirty years I have been brought to concur with these views, though I once thought differently. The convergency of series is a part of Algebra which very few

mathematicians really understand until they reach a much later stage in their reading ; and to ask the immature schoolboy mind to grapple with all their inherent difficulties before being allowed to make any use of the binomial theorem for any exponent, or of the expansions for e^x and $\log_e(1+x)$, in some of their easy and useful applications, seems to me unwise and unpractical. Be this as it may, I venture to think that the examples and illustrations which I have given on pages 489–494, and in Chap. XLIII., will furnish useful matter for numbers of pupils whose algebraical work will never go far beyond the limits of this book, and who in no circumstances would ever find a profitable study in a completely logical treatment of infinite series.

It would have been easy to give incomplete, though plausible, proofs of the theorems in question, but it would have been at variance with the spirit of the times. There is a growing feeling that it is better to give results without proof rather than to offer proofs, in which all the difficulties are glossed over, and which afterwards have to be abandoned as unsound.

H. S. HALL.

Feb. 1912.

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PART III

CHAPTER XXXIX

PERMUTATIONS AND COMBINATIONS

496. EACH of the *groups* or *selections* which can be made by taking some or all of a number of things (without regard to the *order* of the things in each group) is called a **combination**.

Thus the combinations which can be made by taking the four letters a, b, c, d two at a time are six in number: namely,

$ab, ac, ad, bc, bd, cd,$

each of these presenting a different *selection* of two letters.

If the things in each selection are arranged in all possible orders, each of such *arrangements* is called a **permutation**.

Thus each of the foregoing *selections* of the letters a, b, c, d , taken two at a time, admits of two *arrangements*; hence the *permutations* of these letters two at a time are twelve in number: namely,

$ab, ac, ad, bc, bd, cd,$
 $ba, ca, da, cb, db, dc,$

each of these presenting a different *arrangement* of two letters.

Again, a single *combination* of three letters, such as abc , admits of the following *arrangements*:

$abc, acb, bca, bac, cab, cba,$

and so gives rise to *six different permutations*.

497. More generally: when r things are selected out of n , each *selection* is called an **r-combination**, and the number of ways in which such a selection can be made is called *the number of combinations of n things r at a time*, and is denoted by the symbol nC_r .

Each *arrangement* which can be made by taking r things out of n is called an **r-permutation**, and the number of ways in which such an arrangement can be made is called *the number of permutations of n things r at a time*, and is denoted by the symbol nP_r .

498. If we were required to write down all the 3-permutations of 4 letters a, b, c, d , we might proceed as follows: Take a and write each of the other 3 letters after it; we thus obtain three 2-permutations in which a stands first. Similarly, there are three in which b stands first, and so on. Hence the total number of 2-permutations is 4×3 , or 12. Next take *any one* of these, such as ab , and write each of the other letters after it. We thus obtain abc, abd ; that is, from *any one* of the twelve 2-permutations we can obtain two 3-permutations. Hence the total number of 3-permutations is 12×2 , or 24.

499. It is obvious that it would often be a laborious task to find the number of combinations or permutations in any given case by writing them all down exhaustively.

For example, the number of 3-combinations that can be formed out of 10 things would be found to be 120; that is, ${}^{10}C_3 = 120$.

And the number of 4-permutations that can be formed out of 8 things would be found to be 1680; that is, ${}^8P_4 = 1680$.

Hence it is necessary to find general formulæ, in terms of n and r , from which the values of nC_r and nP_r can be readily calculated in every case.

Further, since the order of thought suggests *selection* of groups followed by *arrangement* of the things in each group, it would seem natural to consider combinations first and then to deal with permutations. But it happens that many cases dealing with permutations can be dealt with very simply, by common sense reasoning, by means of an important principle which we shall now explain and illustrate.

500. *If one operation can be performed in m ways, and (when it has been performed in any one of these ways) a second operation can then be performed in n ways, the number of ways of performing the two operations will be $m \times n$.*

If the first operation be performed in *any one* way, we can associate with this any of the n ways of performing the second operation: and thus we shall have n ways of performing the two operations without considering more than *one* way of performing the first; and so, corresponding to *each* of the m ways of performing the first operation, we shall have n ways of performing the two; hence altogether the number of ways in which the two operations can be performed is represented by the product $m \times n$.

EXAMPLE. *There are 10 steamers plying between Liverpool and Dublin; in how many ways can a man go from Liverpool to Dublin and return by a different steamer?*

There are *ten* ways of making the first passage; and with each of these there is a choice of *nine* ways of returning (since the man is not to come back by the same steamer); hence the number of ways of making the two journeys is 10×9 , or 90.

501. The same principle can be used when there are more than two operations, each of which can be performed in a given number of ways.

EXAMPLE 1. *In how many ways can 3 of the letters A, B, C, a, b, c, d be arranged in a row, using only one capital and two small letters, so that the capital always stands first?*

The first place can be filled up in 3 ways, since any one of the capitals may be used. And when the first place has been filled up in any one of these ways, the second place can be filled up in 4 ways, since any one of the letters *a, b, c, d* may be used. And since each way of filling up the first place can be associated with each way of filling up the second, the number of ways of filling up the first two places is given by the product 3×4 . And when the first two places have been filled in any one of these 12 ways, the third place can be filled up by using any of the 3 remaining small letters. Hence, reasoning as before, the number of ways in which the 3 places can be filled up is 12×3 , or 36.

EXAMPLE 2. *Four persons enter a railway carriage in which there are six vacant seats; in how many ways can they take their places?*

The first person may seat himself in 6 ways; and then the second person in 5; the third in 4; and the fourth in 3; and since each of these ways may be associated with each of the others, the required answer is $6 \times 5 \times 4 \times 3$, or 360.

EXAMPLES XXXIX. a 14-15 Nov. 55

1. A field has 4 gates; in how many ways is it possible to enter the field by one gate and come out at another?

2. In how many ways can one consonant and one vowel be chosen out of the letters of the word *Cambridge*?

3. In how many ways is it possible to take one apple, one orange, and one pear from a basket containing 6 apples, 4 oranges, and 5 pears?

4. In how many ways can two prizes be given to a class of 12 boys, (i) if one boy may receive both; (ii) if no boy can receive more than one prize?

5. There are 10 competitors in a race for 3 prizes; in how many ways can the prizes be given?

6. How many different signals can be made by hoisting 4 flags on one mast, with 7 flags to choose from?

7. How many integral numbers can be formed with the four digits 1, 2, 3, 4? How many if 0 is substituted for 1?

8. In how many ways can the letters of the word *minus* be arranged? How many of these will begin with *m*? How many will *not* begin with *m*? How many will begin with *m* and end with *s*?

9. A man lives within reach of 2 boys' schools and 3 girls' schools: in how many ways can he send his 3 sons and 2 daughters to school?

Permutations of Unlike Things

502. To find the number of permutations of n unlike things taken r at a time.

This is the same thing as finding the number of ways in which we can fill up r blank places when we have n unlike things at our disposal.

The first place may be filled up in n ways, for any one of the n things may be taken; when it has been filled up in any one of these ways, the second place can then be filled up in $n-1$ ways; and since each way of filling up the first place can be associated with each way of filling up the second, the number of ways in which the first two places can be filled up is given by the product $n(n-1)$. And when the first two places have been filled up in any one of these ways, the third place can be filled up in $n-2$ ways. And reasoning as before, the number of ways in which three places can be filled up is $n(n-1)(n-2)$.

Proceeding thus, and noticing that at any stage the number of factors is the same as the number of places filled up, we shall have the number of ways in which r places can be filled up equal to

$$n(n-1)(n-2) \dots \text{to } r \text{ factors};$$

and the r^{th} factor is $n-(r-1)$, or $n-r+1$.

Therefore the number of permutations of n things taken r at a time is

$$n(n-1)(n-2) \dots (n-r+1).$$

COR. The number of permutations of n things taken all at a time is

$$n(n-1)(n-2) \dots \text{to } n \text{ factors,}$$

or

$$n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1.$$

503. The product of the first n consecutive numbers is denoted by the symbol $|n$, or $n!$. Either symbol is read “factorial n .”

We have thus proved the two following formulæ:

$$(i) {}^n P_r = n(n-1)(n-2) \dots (n-r+1),$$

$$(ii) {}^n P_n = |n, \text{ or } n!.$$

NOTE. It should be noticed that the suffix r in the symbol ${}^n P_r$ always indicates the number of factors in the formula we are using.

EXAMPLE 1. How many different four-figure numbers can be formed
(i) with the digits 1, 3, 5, 7, 9; (ii) with the digits 0, 1, 3, 5, 7, 9, no digit being used more than once in each number?

(i) We have 5 different things and we have to find the number of permutations of them 4 at a time.

$$\therefore \text{the required number} = {}^5 P_4 = 5 \cdot 4 \cdot 3 \cdot 2 = 120.$$

(ii) Since a number cannot begin with 0, the first place can only be filled up in 5 ways. Then the number of ways of arranging 3 out of the remaining 5 digits is ${}^5 P_3$.

$$\therefore \text{the required number} = 5 \times {}^5 P_3 = 5 \times 5 \cdot 4 \cdot 3 = 300.$$

EXAMPLE 2. In how many ways can the letters of the word *courage* be arranged, if the vowels are always to occupy the odd places?

The 4 vowels can occupy the 4 odd places in $\underline{4}$ ways. The 3 consonants can occupy the 3 even places in $\underline{3}$ ways.

Each arrangement of vowels can be associated with each arrangement of consonants;

$$\therefore \text{the required number} = \underline{4} \times \underline{3} = 4 \cdot 3 \cdot 2 \times 3 \cdot 2 = 144.$$

EXAMPLES XXXIX. b

1. Find the numerical values of $\underline{7}$, 5P_5 , 7P_5 , $8!$, 9P_4 , $\frac{\underline{9}}{\underline{3}}$.
2. How many arrangements can be made by taking (i) five, (ii) all of the letters of the word *number*?
3. If ${}^nP_4 = 18 \times {}^{n-1}P_2$, find n .
4. How many changes can be rung with 5 bells? How many of these will begin with one particular bell?
5. Using each digit only once in each number, find how many numbers between 2000 and 3000 can be formed with the digits 1, 2, 3, 4, 5, 6.
6. How many permutations are there of the letters of the word *orange*, (i) beginning with *o*, (ii) not beginning with *o*?
7. In how many ways can 4 boys and 3 girls be arranged alternately, with a boy at each end of the row?
8. Of the permutations of the letters of the word *factor* taken all together, how many do not begin with *fa*?
9. How many arrangements of the letters of the word *fragile* can be made, if the vowels are always to occupy the first, the last, and the middle places?
10. An examination consists of six papers, of which two are in mathematics. In how many ways can the papers be given out so that the mathematical papers are not consecutive?
11. Shew by general reasoning that ${}^{n+1}P_{r+1} = (n+1) \times {}^nP_r$.
12. If 8 men and 5 women apply for 5 different situations, 3 of which must be filled by men and 2 by women, in how many ways can the situations be filled?
13. There are 8 different situations vacant, of which 3 must be held by men and 2 by women; the remaining 3 may be held by either men or women. If 10 men and 5 women present themselves as candidates, in how many ways can the situations be filled?
14. Find the number of ways in which 6 different books can be arranged (i) if 3 specified books are always together, (ii) if the 3 specified are always separated.

Permutations of things not all different

504. The foregoing formulæ apply only when the things considered are *unlike*. When we speak of things being *dissimilar*, *different*, *unlike*, it is assumed that they are *visibly unlike*, so as to be easily distinguished from each other. Things are considered to be *alike* when they cannot be so distinguished from each other.

The following example should be very carefully studied.

EXAMPLE. *How many different words can be formed with the letters a, d, e, d, d, b, using all the six letters in each word?*

Let x be the required number of words; then if in *any one* of these words (e.g. *debadd*) we were to replace the letters d by new unlike letters different from any of the rest, *from this single word* we could form $\lfloor 3$ new words. If a similar change were made in each of the x words, we should obtain $x \times \lfloor 3$ new words. But since the 6 letters have now become all different the number of words must be equal to $\lfloor 6$;

$$\therefore x \times \lfloor 3 = \lfloor 6, \text{ or } x = \frac{\lfloor 6}{\lfloor 3} = 6 \cdot 5 \cdot 4 = 120.$$

505. *To find the number of ways in which n things may be arranged among themselves, taking them all at a time, when p of the things are alike of one kind, q of them alike of another kind, r of them alike of a third kind, and the rest all different.*

Let there be n letters; suppose p of them to be a , q of them to be b , r of them to be c , and the rest to be unlike.

Let x be the required number of permutations; then if the p letters a were replaced by p unlike letters different from any of the rest, from *any one* of the x permutations, without altering the position of any of the remaining letters, we could form $\lfloor p$ new permutations. Hence if this change were made in each of the x permutations, we should obtain $x \times \lfloor p$ permutations.

If in each of *these* permutations the q letters b were replaced by q unlike letters, the number of permutations would be $x \times \lfloor p \times \lfloor q$.

In like manner, by now replacing the r letters c by r unlike letters, we should finally obtain $x \times \lfloor p \times \lfloor q \times \lfloor r$ permutations.

But the things are now all different, and therefore admit of $\lfloor n$ permutations among themselves. Hence

$$x \times \lfloor p \times \lfloor q \times \lfloor r = \lfloor n;$$

that is,

$$x = \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}, \text{ or } \frac{n!}{p! q! r!};$$

which is the required number of permutations.

Any case in which the things are not all different may be treated similarly.

EXAMPLE 1. *How many different permutations can be made out of the letters of the word assassination taken all together?*

We have 13 letters, of which 4 are *s*, 3 are *a*, 2 are *i*, and 2 are *n*. Hence the number of permutations

$$= \frac{13!}{4!3!2!2!} = 13 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 3 \cdot 5$$

$$= 1001 \times 10800 = 10810800.$$

EXAMPLE 2. *How many numbers can be formed by using the digits 1, 2, 5, 4, 5, 5, 6, 1, 8, so that the odd digits always occupy the odd places?*

(i) The odd digits, 1, 5, 5, 5, 1, can be arranged in the five odd places in $\frac{5!}{3!2!}$, or 10 ways.

(ii) The even digits, 2, 4, 6, 8, can be arranged in the four even places in $4!$, or 24 ways.

Each of the ways in (i) can be associated with each of the ways in (ii).

Hence the required number $= 10 \times 24$, or 240.

[Examples XXXIX. c. 1-9, page 464, may be taken here.]

Permutations of things which may be repeated

506. *To find the total number of r -permutations of n different things when each thing may be repeated up to r times in any arrangement.*

Consider n different kinds of letters $a, b, c, \dots$, not less than r of each kind. Then the required number of permutations will be equal to the number of ways in which r places can be filled up by using *any one* of these letters in each place.

The first place may be filled up in n ways, and, when it has been filled up in any one way, the second place may also be filled up in n ways, since we may use the same thing again. Therefore the number of ways in which the first two places can be filled up is $n \times n$ or n^2 .

The third place can also be filled up in n ways, and therefore the first three places in n^3 ways, and so on.

Since at any stage the index of n is always the same as the number of places filled up, we shall have the number of ways in which the r places can be filled up equal to n^r .

EXAMPLE. *In how many ways can 5 prizes be given away to 4 boys, when each boy is eligible for all the prizes?*

Any one of the prizes can be given in 4 ways; and then any one of the remaining prizes can also be given in 4 ways, since it may be obtained by the boy who has already received a prize. Thus two prizes can be given away in 4^2 ways, three prizes in 4^3 ways, and so on. Hence the 5 prizes can be given away in 4^5 or 1024 ways.

EXAMPLES XXXIX. c

[In Examples 1–9 each thing occurs once only in any arrangement.]

1. Find the number of permutations which can be made by using all the letters of the following words :

- (i) *tobacco* ; (ii) *alluvial* ;
(iii) *tittle-tattle* ; (iv) *appropriation*.

In (i) and (iii) how many of the permutations begin with *t* ?

2. Among the permutations of the letters of the word *series* how many begin and end with *s* ? In how many are the vowels and consonants placed alternately ?

3. How many different numbers can be formed by using the seven digits 2, 3, 4, 3, 3, 1, 2 ? How many with the digits 2, 3, 4, 3, 0, 2 ?

4. Without assuming the general formula, find the number of permutations of all the letters of the word *zoological*.

5. In how many ways can the letters of the word *cannon* be arranged (i) if the two vowels always come together ; (ii) if the relative position of the vowels and consonants is not altered ?

6. I have 2 exactly similar copies of Algebra, 3 of Geometry, and single copies of Arithmetic and Trigonometry. In how many ways can these books be distributed among 7 boys, one volume to each ?

7. How many *even* numbers, each of 7 digits, can be formed with the digits 3, 2, 5, 4, 3, 5, 5 ?

8. In how many ways can 2 sixes, 3 fives, and 5 twos be thrown with 10 dice ?

9. How many numbers greater than 30,000 can be made by using all the digits 1, 4, 4, 3, 5 ?

(Permutations with repetitions.)

10. Find the total number of ways in which 5 sparrows can perch on 3 trees when there is no restriction as to the choice of tree. In how many of these ways will one particular sparrow be alone on a tree ?

11. How many 4-permutations can be made out of the letters *a, b, e, f, g, k* when repetitions are allowed ?

12. In how many ways can I make 4 journeys with 3 conveyances to choose from ?

13. A letter-lock consists of 4 rings each marked with 5 different letters ; how many unsuccessful attempts can be made to open the lock ?

14. In how many ways can 5 hats be divided between 2 men ?

15. In how many ways can a man harness 3 beasts to a plough when he has horses, oxen, mules, and asses to choose from ?

16. Shew that the total number of permutations (with repetitions) of n different things, not more than r being taken at a time, is $\frac{n(n^r - 1)}{n - 1}$.

Combinations of Unlike Things

507. To find the number of r -combinations of n unlike things.
[See Art. 497.]

Let nC_r denote the required number of combinations, or groups. If the r things in each group are arranged in all possible ways, each group will give rise to $|r$ arrangements.

$$\begin{aligned} \therefore {}^nC_r \times |r &\text{ is equal to the number of } r\text{-permutations of } n \text{ things;} \\ \text{hence } {}^nC_r \times |r &= {}^nP_r = n(n-1)(n-2) \dots (n-r+1); \\ \therefore {}^nC_r &= \frac{n(n-1)(n-2) \dots (n-r+1)}{|r} \dots \dots \dots (1) \end{aligned}$$

This formula for nC_r may be written in a different form; for if we multiply above and below by $|n-r$, we obtain

$$\frac{n(n-1)(n-2) \dots (n-r+1) \times |n-r}{|r|n-r};$$

and since $n(n-1)(n-2) \dots (n-r+1) \times |n-r = |n$, we have

$${}^nC_r = \frac{|n}{|r|n-r}, \quad \text{or} \quad \frac{n!}{r!(n-r)!} \dots \dots \dots (2)$$

NOTE.—In using formula (1) it is useful to remember that the suffix in the symbol nC_r denotes the number of factors in both numerator and denominator.

EXAMPLE. From 12 books in how many ways can a selection of 5 be made, (1) when one specified book is always included, (2) when one specified book is always excluded?

(1) Since the specified book is to be included in every selection, we have only to choose 4 out of the remaining 11.

$$\text{Hence the number of ways} = {}^{11}C_4 = \frac{11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4} = 330.$$

(2) Since the specified book is always to be excluded, we have to select the 5 books out of the remaining 11.

$$\text{Hence the number of ways} = {}^{11}C_5 = \frac{11 \times 10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4 \times 5} = 462.$$

508. The number of combinations of n things r at a time is equal to the number of combinations of n things $n-r$ at a time.

In making all the possible combinations of n things, to each group of r things we select, there is left a corresponding group of $n-r$ things; that is, the number of combinations of n things r at a time is the same as the number of combinations of n things $n-r$ at a time;

$$\therefore {}^nC_r = {}^nC_{n-r}.$$

Thus

$${}^{15}C_{13} = {}^{15}C_2 = \frac{15 \cdot 14}{1 \cdot 2} = 105.$$

EXAMPLE. To prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

nC_r = the number of r -combinations of $(n+1)$ things when one specified thing is always *excluded*.

${}^nC_{r-1}$ = the number of r -combinations of $(n+1)$ things when the specified thing is always *included*.

The sum of these = the total number of r -combinations of $(n+1)$ things ;

$$\therefore {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r.$$

509. In the following examples the first thing to decide is whether the conditions imply *arrangements* or *selections only*.

When arrangements are involved a formula for *permutations* must not be used until suitable *selections* have been made according to the conditions of the question.

EXAMPLE 1. From 7 masters and 4 boys a committee of 6 is to be formed : in how many ways can this be done, (i) when the committee contains exactly 2 boys, (ii) at least 2 boys ?

Here we are not concerned with the possible arrangements of the members of the committee amongst themselves. Hence it is a case of *selections only*.

(i) The number of ways in which the 2 boys can be chosen is 4C_2 ; and the number of ways in which the 4 masters can be chosen is 7C_4 . Each group of boys can be combined with each group of masters ;

$$\therefore \text{the required number} = {}^4C_2 \times {}^7C_4 = \frac{4 \cdot 3}{1 \cdot 2} \times \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 210.$$

(ii) All the suitable combinations will be found by forming all the groups containing 2 boys and 4 masters ; then 3 boys and 3 masters ; and, lastly, 4 boys and 2 masters.

The *sum* of these results will give the answer. Hence the required number of ways = ${}^4C_2 \times {}^7C_4 + {}^4C_3 \times {}^7C_3 + {}^4C_4 \times {}^7C_2$

$$\begin{aligned} &= \frac{4 \cdot 3}{1 \cdot 2} \times \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} + 4 \times \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} + 1 \times \frac{7 \cdot 6}{1 \cdot 2} \\ &= 210 + 140 + 21 = 371. \end{aligned}$$

EXAMPLE 2. Out of 7 consonants and 4 vowels, how many words can be made each containing 3 consonants and 2 vowels ?

Here each "word" means a different *arrangement*; hence we must first select sets of 3 consonants and 2 vowels, and then combine them into sets of 5 letters which may be arranged among themselves.

The number of ways of choosing the 3 consonants is 7C_3 , and the number of ways of choosing the 2 vowels is 4C_2 ; hence the number of combined groups, each containing 3 consonants and 2 vowels, is ${}^7C_3 \times {}^4C_2$.

Each of these groups contains 5 different letters which may be arranged among themselves in $\underline{5}$ ways ;

$$\begin{aligned} \therefore \text{the required number of words} &= \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} \times \frac{4 \cdot 3}{1 \cdot 2} \times \underline{5} \\ &= 5 \times \underline{7} = 25200. \end{aligned}$$

510. *To find the total number of ways in which it is possible to make a selection by taking some or all of n things.*

Each thing may be dealt with in two ways, for it may be either taken or left; and since either way of dealing with any one thing may be associated with either way of dealing with each of the others, the number of ways of dealing with the n things is

$$2 \times 2 \times 2 \times 2 \dots \text{to } n \text{ factors.}$$

But this includes the case in which all the things are left; therefore rejecting this case, the total number of ways is $2^n - 1$.

This is sometimes referred to as "the total number of combinations of n things."

EXAMPLE. *A man has 6 friends: in how many ways may he invite one or more of them to dinner?*

He has to select some or all of his 6 friends;

$$\therefore \text{the required number of ways} = 2^6 - 1, \text{ or } 63.$$

Or thus: The guests may be invited singly, by twos, threes,; therefore the number of selections $= {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$
 $= 6 + 15 + 20 + 15 + 6 + 1 = 63.$

511. *To find the number of ways in which $m+n$ things can be subdivided into two groups containing m and n things respectively.*

The number of ways of choosing a group of m things is ${}^{m+n}C_m$, and with each of such selections the n things make up the second group.

$$\therefore \text{the required number} = \frac{|m+n|}{|m| |n|}.$$

NOTE. If $n=m$, the groups are equal, and it is possible to interchange the two groups without obtaining a new subdivision; hence the number of different ways of subdivision is $\frac{|2m|}{|m| |m| |2|}.$

EXAMPLE. *In how many ways can 12 books be divided into 3 sets each containing 4 books?*

The books can be divided into sets of 4 and 8 in ${}^{12}C_4$ ways.

Then each set of 8 can be divided into 2 sets of 4 in 8C_4 ways.

Thus the number of ways of making up the 3 sets

$$= \frac{|12|}{|4| |8|} \times \frac{|8|}{|4| |4|} = \frac{|12|}{|4| |4| |4|}, \text{ or } \frac{|12|}{(|4|)^3}.$$

But these are not all *different* modes of subdivision, for since the sets are equal there are $|3|$ different orders in which the sets can be arranged without giving a different subdivision.

$$\text{Hence the required number} = \frac{|12|}{(|4|)^3 \times |3|}.$$

[Examples XXXIX. d. 1-20, page 470, may be taken here.]

512. For a given value of n to find the value of r which makes nC_r greatest.

We have ${}^nC_r = {}^nC_{r-1} \times \frac{n-r+1}{r}$, since nC_r has one more factor than ${}^nC_{r-1}$ both in numerator and denominator.

The multiplying factor $\frac{n-r+1}{r}$ may be written $\frac{n+1}{r} - 1$, which shews that it decreases as r increases. Hence by giving to r the values 1, 2, 3, ... in succession, nC_r is continually increased until $\frac{n+1}{r} - 1$ becomes equal to 1 or less than 1.

Hence ${}^nC_r >$, or $= {}^nC_{r-1}$ according as $\frac{n+1}{r} - 1 >$, or $= 1$;

that is, according as $\frac{n+1}{r} >$, or $= 2$;

that is, according as $\frac{n+1}{2} >$, or $= r$.

Also r can only have integral values.

(i) If n is even, then $\frac{n+1}{2}$ is a fraction, and the greatest value r can have is $\frac{n}{2}$.

Hence nC_r is greatest when $r = \frac{n}{2}$.

(ii) If n is odd, then $\frac{n+1}{2}$ is integral, and when $r = \frac{n+1}{2}$ we have ${}^nC_r = {}^nC_{r-1}$.

Hence nC_r is greatest when $r = \frac{n+1}{2}$, or $\frac{n-1}{2}$; the result being the same in the two cases.

513. To find the total number of ways in which it is possible to make a selection by taking some or all out of $p+q+r+\dots$ things, whereof p are alike of one kind, q alike of a second kind, r alike of a third kind, and so on.

The p things may be disposed of in $p+1$ ways; for we may take 0, 1, 2, 3, ... p of them. Similarly the q things may be disposed of in $q+1$ ways; the r things in $r+1$ ways, and so on.

Hence the number of ways in which all the things may be disposed of is $(p+1)(q+1)(r+1)\dots$.

But this includes the case in which none of the things are taken; therefore, rejecting this case, the total number of ways is

$$(p+1)(q+1)(r+1)\dots - 1.$$

514. Miscellaneous Examples in Permutations and Combinations.

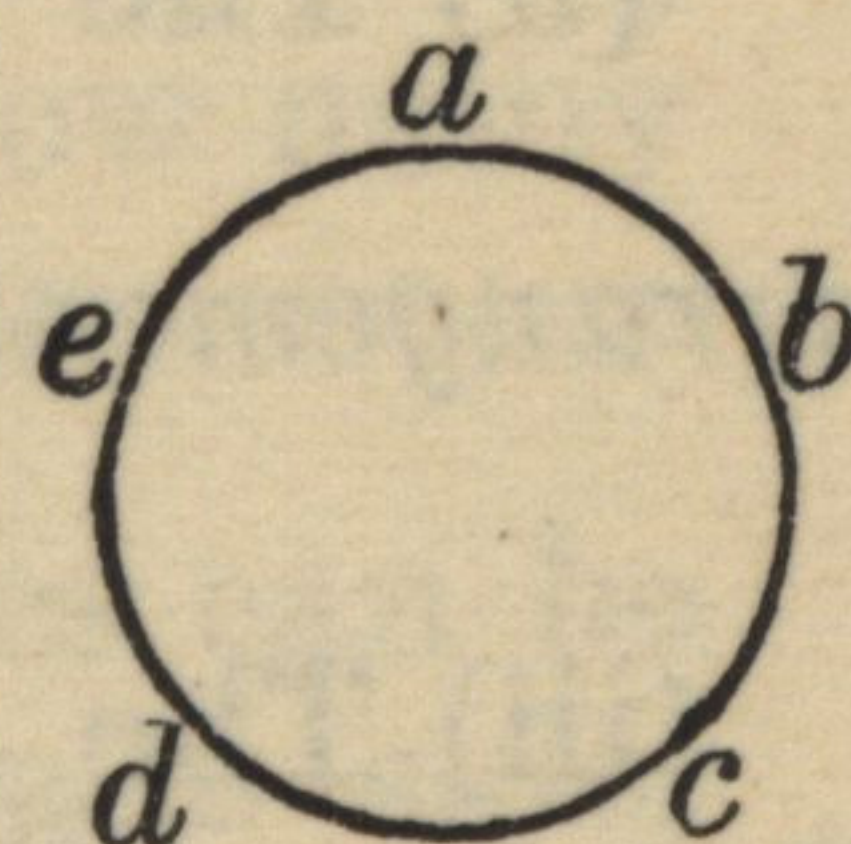
EXAMPLE 1. In how many ways can 5 things, such as a, b, c, d, e, be arranged in a ring?

If we call an arrangement *linear* or *circular* according as the things are placed in a *row* or in a *ring*, it will be seen that while any linear arrangement depends upon the *absolute* position in which the things stand, a circular arrangement depends only on the position of the things *relatively* to each other.

Thus the 5 arrangements

abcde, bcdea, cdeab, deabc, eabcd,

which are all different *linear* arrangements, have no essential difference when regarded as *circular* arrangements. For they are obtained by merely reading the 5 letters in cyclic order, starting from each letter successively. Hence in the case of 5 different things each circular arrangement gives 5 linear arrangements.



Hence the number of circular arrangements

$$= \frac{1}{5} \text{ of the number of linear arrangements}$$

$$= \frac{1}{5} \mid 5 = \mid 4.$$

Similarly n different things can be arranged round a circle in $\mid n - 1$ ways.

We may arrive at the same result briefly as follows: since in each arrangement we are concerned only with the position of a thing relatively to the others, let one thing be placed in *any one* position, then the remaining $n - 1$ things can fill the remaining places in $\mid n - 1$ ways.

EXAMPLE 2. I have 6 sorts of books and 3 of each sort. In how many ways can a selection be made from them?

In the case of each book we may take 0, 1, 2, 3; that is, we may deal with each book in 4 ways, and therefore with the 6 sorts of books in 4^6 ways. But this includes the case in which no selection is made; hence the required number

$$= 4^6 - 1 = 4095.$$

EXAMPLE 3. A railway carriage will accommodate 5 passengers on each side: in how many ways can 10 persons take their seats when two of them decline to face the engine, and a third cannot travel backwards?

Divide the 7 persons who can sit on either side into two groups containing 3 and 4 respectively. This can be done in 7C_3 ways.

Now each side admits of $\mid 5$ arrangements;

$$\therefore \text{the required number} = {}^7C_3 \times \mid 5 \times \mid 5$$

$$= \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} \times 120 \times 120$$

$$= 504000.$$

EXAMPLE 4. Find the number of arrangements that can be made from the letters a, a, a, b, b, c, d, e, taking them 4 at a time.

Here we have 8 letters of 5 different kinds.

In finding groups of 4, these may be classified as follows :

- (i) 3 alike, 1 different ; (ii) 2 alike, 2 others alike ;
 (iii) 2 alike, 2 different ; (iv) all 4 different.

(i) The selection can be made in 4 ways, for each of the letters b, c, d, e may be taken with the group aaa.

$$\therefore \text{the number of arrangements} = 4 \times \frac{4}{1} = 16.$$

(ii) The selection can be made in 1 way only. Hence the number of arrangements = $\frac{4}{2 \times 2} = 6$.

(iii) The selection can be made in $2 \times {}^4C_2$ ways, for we have a choice of 2 pairs of like letters and 2 out of the remaining 4 letters.

$$\therefore \text{the number of arrangements} = 2 \times \frac{4 \cdot 3}{1 \cdot 2} \times \frac{4}{2} = 12 \times 12 = 144.$$

(iv) The selection can be made in 5C_4 , or 5 ways. Hence the number of arrangements = $5 \times 4 = 20$.

$$\therefore \text{the total number of arrangements} = 16 + 6 + 144 + 20 = 186.$$

EXAMPLES XXXIX. d

- Find the numerical values of 6C_3 , 9C_6 , ${}^{11}C_9$, ${}^{50}C_{48}$, $\frac{15!}{13!2!}$.
- In how many ways can 4 boys be chosen out of a form of 21, so as always to include the head of the form ?
- There are 8 bay, 7 black, and 5 roan horses for sale ; in how many ways is it possible to buy 12 horses, 4 of each colour ?
- How many parcels of 6 books may be made out of 7 Latin and 3 Greek books, (i) when there is no restriction ; (ii) when each parcel holds 1 Greek book ; (iii) when each parcel holds all the Greek books ?
- A committee of 6 is to be chosen from 7 Englishmen, 1 Frenchman, 1 German, and 1 Italian. In how many ways can the choice be made, (i) to include the Frenchman ; (ii) to include exactly one foreigner ; (iii) to include at least one foreigner ?

6. If $2 \times {}^nC_4 = 35 \times \frac{n}{2}C_3$, find n . 7. If ${}^{28}C_{r+4} = {}^{28}C_{r-2}$, find r .

8. In the formula ${}^nC_r = \frac{n!}{r!(n-r)!}$ put $r=n$, and hence find a meaning for the symbol $\frac{n!}{n!0!}$.

9. Out of the 26 letters of the alphabet how many words can be made consisting of 4 letters, one of which must be a ?
10. How many words can be made by taking 3 consonants and 2 vowels out of 15 consonants and 4 vowels ?
11. In how many ways can 5 chairs be occupied by 3 men and 2 boys taken from 6 men and 5 boys ? Explain clearly why the formula ${}^6P_3 \times {}^5P_2$ does not give the correct answer.
12. Of 15 men, 3 can steer and cannot row, and the rest can row but cannot steer ; in how many ways can the crew of an eight-oar, with a coxswain, be made up ?
13. There are 4 candidates for 2 vacancies. There are 3 electors, each of whom can vote for two candidates or plump for one : in how many ways can the votes be taken ?
14. From 5 ladies and 7 gentlemen how many different parties can be made up to travel in a railway carriage which has 4 seats on each side, supposing all the ladies to be included in each party ? In how many different ways can the seats be occupied if the corner seats are always to be reserved for ladies ?
15. Find the total number of selections that can be made out of 8 things.
16. How many parcels, each containing not more than 6, can be made with 8 books to choose from ?
17. How many choices has a purchaser from 10 things exposed for sale ?
18. A house has 9 windows in the front : how many different signals can be made by leaving one or more of the windows open ?
19. At an election three districts are to be canvassed by 10, 12, and 8 men respectively. If 30 men volunteer, in how many ways can they be allotted to the different districts ? [*Give the answer in factorials.*]
20. In how many ways can 52 cards be divided, (i) into four sets of 13 each ; (ii) equally among four players ?

(*Miscellaneous.*)

21. In how many ways can 6 letters be placed in 6 envelopes, one in each, if 2 of the letters are too large for one of the envelopes ?
22. A telegraph has 4 arms, and each arm has 3 positions, including the position of rest ; find the total number of signals that can be made.
23. In how many ways can 10 examination papers be arranged so that the best and worst papers never come together ?
24. Find the number of ways in which 5 ladies and 5 gentlemen can be placed alternately in a ring.

25. Find the number of ways in which mn things can be divided equally among n persons.
26. From 4 pears, 2 apples, and 3 oranges how many selections of fruit can be made, taking at least one of each kind ?
27. There are 10 points in a plane, 4 of which are collinear : find the number (i) of straight lines, (ii) of triangles which result from joining them.
28. In how many ways can a crew of 8 be made up when 2 of the men can only row on stroke side, and 1 only on bow side ?
29. Find the number of ways in which n books can be arranged on a shelf so that two specified books are not together.
30. Find the number of ways in which the letters of the word *abstemious* may be arranged, (i) without altering the place of any vowel ; (ii) without changing the order of the vowels.
31. Out of 4 ladies and 5 gentlemen how many sets of two pairs for lawn-tennis can be arranged, (i) when there is no restriction as to the players on each side ; (ii) when each pair consists of one lady and one gentleman ?
32. Find the number of ways in which a selection can be made from m sorts of things, and n things of each sort.
33. Find the number (i) of selections, (ii) of arrangements that can be made by taking 4 letters from the word *zoology*.
34. A man has 8 bachelor friends, and he wishes to invite r of them to dine with him on successive evenings as long as he can have a different selection each time. For how many evenings is it possible to continue these parties, and how often will each of the 8 friends form one of the party ?
35. In how many ways can n men be arranged in a row if two specified men are neither of them to be at either extremity of the row ?
36. If there are 10 things, of which 2 are alike, find the number of permutations of them taken 5 at a time.
37. Find the number (i) of combinations, (ii) of permutations that can be made from the letters of the word *expression*, taken 4 at a time.
38. How many choices are there in buying books from a book-stall where 5 copies of one book, 6 copies of another book, and single copies of 5 other different books are offered for sale ?
39. Twenty-two men arrange to play a cricket match. If two of the men are brothers, shew that the number of ways in which the teams can be made up so that the brothers do not play on the same side is $2 \mid 19 \div \mid 10 \mid 9$.

26 Jan. 56

CHAPTER XL

MATHEMATICAL INDUCTION

515. IN proving the truth of certain mathematical results or formulæ, it is often convenient to use an indirect method known as **Mathematical Induction**. We shall explain this method of proof by examples.

EXAMPLE. To prove that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$ to n terms $= \frac{n}{n+1}$.

We can easily shew that this formula is true in simple cases, such as when $n=1$, or 2, or 3. We wish, however, to prove it true in *all* cases.

Assume that it is true when n terms are taken ;

that is, $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

To each side add $\frac{1}{(n+1)(n+2)}$, which is the $(n+1)^{\text{th}}$ or *next* term in the series ; then

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots \text{ to } (n+1) \text{ terms} &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} \left(n + \frac{1}{n+2} \right) \text{ OK} \\ &= \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} = \frac{n^2 + 2n + 1}{(n+1)(n+2)} = \frac{n+1}{n+2} \\ &= \frac{n+1}{(n+1)+1} \text{ OK} \end{aligned}$$

Now this last result is of the same form as that assumed for n terms, with $n+1$ written in the place of n . In other words, if the result is true when we take a certain number of terms, whatever that number may be, it is true when we increase that number by one ; but by trial it is true when 3 terms are taken ; hence it is true for 4 terms ; therefore for 5 terms ; and so on. Thus the result is true universally.

The method of proof involves the following steps :

(i) Assuming the truth of the formula in *any one case* (say the n^{th}), we shew that it must be true in the *next* case, viz. the $(n+1)^{\text{th}}$.

(ii) By trial we shew that it is true in a certain simple case ; hence it is true in the case next after that verified ; hence it is true in the next case ; and so on. Hence it is true in all cases.

516. As another example we will prove one of the theorems relating to divisibility given in Art. 468.

EXAMPLE. *Shew that $x^n - y^n$ is divisible by $x - y$ for all positive integral values of n .*

By going through one step of division we have

$$\frac{x^n - y^n}{x - y} = x^{n-1} + \frac{y(x^{n-1} - y^{n-1})}{x - y}.$$

If therefore $x^{n-1} - y^{n-1}$ is divisible by $x - y$, then $x^n - y^n$ is also divisible by $x - y$.

But $x^2 - y^2$ is divisible by $x - y$; therefore $x^3 - y^3$ is divisible by $x - y$; therefore $x^4 - y^4$ is divisible by $x - y$; and so on. Hence the proposition is universally true.

517. Speaking generally, it will be seen that proof by induction can be conveniently applied to all theorems which admit of successive cases corresponding to the order of the natural numbers 1, 2, 3, ... n . But if the truth of a general theorem depends upon whether n is odd or even, it must be remembered that in considering *any one* case (say the n^{th}), the *next* case will be the $(n+2)^{\text{th}}$.

EXAMPLES XL

Prove by Induction :

$$1. \quad a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}.$$

$$2. \quad 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$3. \quad 1^3 + 2^3 + 3^3 + \dots + n^3 = \left\{ \frac{n(n+1)}{2} \right\}^2.$$

$$4. \quad 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) = \frac{1}{3}n(n+1)(n+2).$$

$$5. \quad 3 + 3^2 + 3^3 + \dots + 3^n = \frac{3(3^n - 1)}{2}.$$

$$6. \quad \frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \dots \text{ to } n \text{ terms} = \frac{n}{3n+1}.$$

7. Prove by Induction :

- (i) that $x^n + y^n$ is divisible by $x + y$ when n is any odd positive integer ;
- (ii) that $x^n - y^n$ is divisible by $x + y$ when n is any even positive integer.

27 April 1948

CHAPTER XLI

THE BINOMIAL THEOREM

518. It may be shewn by actual multiplication that

$$\begin{aligned} & (x+a)(x+b)(x+c)(x+d) \\ &= x^4 + (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2 \\ &+ (abc+abd+acd+bcd)x + abcd. \dots\dots\dots(1) \end{aligned}$$

We may, however, write down this result by inspection; for the complete product consists of the sum of a number of partial products each of which is formed by multiplying together four letters, *one* being taken from *each* of the four factors. If we examine the way in which the various partial products are formed, we see that

(i) the term x^4 is formed by taking the letter x out of *each* of the factors;

(ii) the terms involving x^3 are formed by taking the letter x out of *any three* factors, in every way possible, and *one* of the letters a, b, c, d out of the remaining factor;

(iii) the terms involving x^2 are formed by taking the letter x out of *any two* factors, in every way possible, and *two* of the letters a, b, c, d out of the remaining factors;

(iv) the terms involving x are formed by taking the letter x out of *any one* factor, and *three* of the letters a, b, c, d out of the remaining factors;

(v) the term independent of x is the product of all the letters a, b, c, d .

EXAMPLE. Find the value of $(x-2)(x+3)(x-5)(x+9)$.

The product

$$\begin{aligned} &= x^4 + (-2+3-5+9)x^3 + (-6+10-18-15+27-45)x^2 \\ &\quad + (30-54+90-135)x + 270 \\ &= x^4 + 5x^3 - 47x^2 - 69x + 270. \end{aligned}$$

519. If in equation (1) of the preceding article we suppose $b=c=d=a$, we obtain

$$(x+a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4.$$

We shall now employ the same method to prove a formula known as the **Binomial Theorem**, by which any binomial of the form $x+a$ can be raised to any assigned positive integral power.

Substituting in full for ${}^nC_1, {}^nC_2, {}^nC_3, \dots$ we obtain

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1 \cdot 2} a^2 x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^3 x^{n-3} + \dots$$

$$+ \frac{n(n-1)(n-2) \dots (n-r+1)}{\underbrace{1 \cdot 2 \cdot 3 \dots r}_r} a^r x^{n-r} + \dots + a^n.$$

This is the *Binomial Theorem*, and the expression on the right is said to be the expansion of $(x+a)^n$.

521. Proof by Induction. We have to prove that

$$(x+a)^n = x^n + {}^nC_1 ax^{n-1} + {}^nC_2 a^2 x^{n-2} + \dots + {}^nC_r a^r x^{n-r} + \dots + a^n.$$

By actual multiplication or involution, we have

$$(x+a)^2 = x^2 + 2ax + a^2 = x^2 + {}^2C_1 ax + a^2,$$

$$(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3 = x^3 + {}^3C_1 ax^2 + {}^3C_2 a^2x + a^3.$$

Assume that the formula is true for the n^{th} power of $(x+a)$; that is,

$$(x+a)^n = x^n + {}^nC_1 ax^{n-1} + {}^nC_2 a^2 x^{n-2} + \dots + {}^nC_r a^r x^{n-r} + \dots + a^n.$$

Multiply both sides by $x+a$; then we have

$$x(x+a)^n = x^{n+1} + {}^nC_1 ax^n + {}^nC_2 a^2 x^{n-1} + \dots + {}^nC_r a^r x^{n-r+1} + \dots + a^n x,$$

$$a(x+a)^n = ax^n + {}^nC_1 a^2 x^{n-1} + \dots + {}^nC_{r-1} a^{r-1} x^{n-r+1} + \dots + a^{n+1}.$$

Therefore by addition, after collecting like terms on the right,

$$(x+a)^{n+1} = x^{n+1} + ({}^nC_1 + 1)ax^n + ({}^nC_2 + {}^nC_1)a^2 x^{n-1} + \dots$$

$$+ ({}^nC_r + {}^nC_{r-1})a^r x^{n-r+1} + \dots + a^{n+1}.$$

Now

$${}^nC_r + {}^nC_{r-1} = \frac{\overbrace{1 \cdot 2 \cdot 3 \dots r}^n}{\overbrace{1 \cdot 2 \cdot 3 \dots r}^r \overbrace{1 \cdot 2 \cdot 3 \dots r-1}^{n-r}} + \frac{\overbrace{1 \cdot 2 \cdot 3 \dots r-1}^{n-1}}{\overbrace{1 \cdot 2 \cdot 3 \dots r-1}^{r-1} \overbrace{1 \cdot 2 \cdot 3 \dots r-2}^{n-r+1}}$$

$$= \frac{\overbrace{1 \cdot 2 \cdot 3 \dots r-1}^{n-1}}{\overbrace{1 \cdot 2 \cdot 3 \dots r-1}^{r-1} \overbrace{1 \cdot 2 \cdot 3 \dots r-2}^{n-r+1}} \left(\frac{1}{r} + \frac{1}{n-r+1} \right)$$

$$= \frac{\overbrace{1 \cdot 2 \cdot 3 \dots r-1}^{n-1}}{\overbrace{1 \cdot 2 \cdot 3 \dots r-1}^{r-1} \overbrace{1 \cdot 2 \cdot 3 \dots r-2}^{n-r+1}} \cdot \frac{n+1}{r(n+1-r)} = \frac{\overbrace{1 \cdot 2 \cdot 3 \dots r}^{n+1}}{\overbrace{1 \cdot 2 \cdot 3 \dots r}^r \overbrace{1 \cdot 2 \cdot 3 \dots r-1}^{n+1-r}}$$

$$= {}^{n+1}C_r.$$

$\therefore$ also ${}^nC_1 + 1 = {}^{n+1}C_1$, ${}^nC_2 + {}^nC_1 = {}^{n+1}C_2$, and so on.

$$\therefore (x+a)^{n+1} = x^{n+1} + {}^{n+1}C_1 ax^n + {}^{n+1}C_2 a^2 x^{n-1} + \dots$$

$$+ {}^{n+1}C_r a^r x^{n+1-r} + \dots + a^{n+1}.$$

Thus the terms of $(x+a)^{n+1}$ are of the same form as those assumed for the expansion of $(x+a)^n$, with $n+1$ taking the place of n . Hence, if the theorem is true for the n^{th} power of $x+a$, it is also true for the $(n+1)^{\text{th}}$ power. But we have seen that it is true for $(x+a)^3$; therefore it is true for $(x+a)^4$; therefore for $(x+a)^5$, and so on. Thus the theorem is true for any positive integral value of n .

522. In the formula

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1 \cdot 2} a^2 x^{n-2} + \dots$$

$$+ \frac{n(n-1)(n-2) \dots (n-r+1)}{1 \cdot 2 \cdot 3 \dots r} a^r x^{n-r} + \dots + a^n,$$

the following points should be noted :

- (i) The number of terms on the right is $n+1$; that is, one more than the index of $x+a$ on the left.
- (ii) The index of a in any term is the last factor in the denominator of the coefficient of that term.
- (iii) In every term the sum of the indices of x and a is n .

523. We shall often quote the theorem in the form

$$(x+a)^n = x^n + {}^nC_1 ax^{n-1} + {}^nC_2 a^2 x^{n-2} + \dots + {}^nC_r a^r x^{n-r} + \dots + {}^nC_n a^n.$$

If we write $-a$ for a , we obtain

$$(x-a)^n = x^n + {}^nC_1 (-a)x^{n-1} + {}^nC_2 (-a)^2 x^{n-2} + {}^nC_3 (-a)^3 x^{n-3} + \dots$$

$$+ {}^nC_r (-a)^r x^{n-r} + \dots + {}^nC_n (-a)^n$$

$$= x^n - {}^nC_1 ax^{n-1} + {}^nC_2 a^2 x^{n-2} - {}^nC_3 a^3 x^{n-3} + \dots + {}^nC_n (-a)^n.$$

Thus the terms of $(x+a)^n$ and $(x-a)^n$ are *numerically* the same, but in $(x-a)^n$ they are alternately positive and negative, and the last term is positive or negative according as n is even or odd.

NOTE. For the sake of uniformity we may use nC_0 as the coefficient of x^n . This is justified by the consideration that there is *only one* way of making *no* selection (*i.e.* rejecting all) out of n things. Hence ${}^nC_0 = 1$.

EXAMPLE 1. Find the expansion of $(x+3y)^6$.

Putting $3y$ for a in the formula, we have

$$\text{the expansion} = x^6 + {}^6C_1 x^5 (3y) + {}^6C_2 x^4 (3y)^2 + {}^6C_3 x^3 (3y)^3 + {}^6C_4 x^2 (3y)^4$$

$$+ {}^6C_5 x (3y)^5 + {}^6C_6 (3y)^6.$$

Hence by calculating the values of ${}^6C_1, {}^6C_2, {}^6C_3 \dots$ we have

$$(x+3y)^6 = x^6 + 6 \cdot 3x^5y + 15 \cdot 9x^4y^2 + 20 \cdot 27x^3y^3 + 15 \cdot 81x^2y^4 + 6 \cdot 243xy^5 + 729y^6$$

$$= x^6 + 18x^5y + 135x^4y^2 + 540x^3y^3 + 1215x^2y^4 + 1458xy^5 + 729y^6.$$

EXAMPLE 2. Find the expansion of $(a-2x)^7$.

$$(a-2x)^7 = a^7 - {}^7C_1 a^6 (2x) + {}^7C_2 a^5 (2x)^2 - {}^7C_3 a^4 (2x)^3 + \dots \text{to 8 terms.}$$

Now remembering that ${}^nC_r = {}^nC_{n-r}$, after calculating the coefficients up to 7C_3 , the rest may be written down at once ; for ${}^7C_4 = {}^7C_3$; ${}^7C_5 = {}^7C_2$; and so on. Hence

$$(a-2x)^7 = a^7 - 7a^6(2x) + \frac{7 \cdot 6}{1 \cdot 2} a^5 (2x)^2 - \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} a^4 (2x)^3 + \dots$$

$$= a^7 - 7a^6(2x) + 21a^5(2x)^2 - 35a^4(2x)^3 + 35a^3(2x)^4$$

$$- 21a^2(2x)^5 + 7a(2x)^6 - (2x)^7$$

$$= a^7 - 14a^6x + 84a^5x^2 - 280a^4x^3 + 560a^3x^4$$

$$- 672a^2x^5 + 448ax^6 - 128x^7.$$

524. The symbols ${}^nC_1, {}^nC_2, {}^nC_3, \dots$ are often referred to as the **Binomial Coefficients of the n^{th} order**, and it is important to be able to calculate their numerical values quickly. We shall now explain a rule by which this may be done.

In $(x+a)^1$ the coefficients are 1 1,
„ $(x+a)^2$ „ „ 1 2 1,
„ $(x+a)^3$ „ „ 1 3 3 1,
„ $(x+a)^4$ „ „ 1 4 6 4 1.

Here the first and last coefficient in each line is 1, and for the rest we notice that in any line each coefficient is equal to the coefficient immediately above it added to the preceding coefficient in the same line.

Thus in the 3rd line $3=2+1, \quad 3=1+2$;
and in the 4th line $4=3+1, \quad 6=3+3, \quad 4=1+3$.

The general rule is given in the Example of Art. 508, where it is shewn that ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$, a result which may be verbally expressed by saying that *the r^{th} binomial coefficient of any order is equal to the sum of the r^{th} and $(r-1)^{\text{th}}$ coefficients of the preceding order.*

By the use of this rule it is easy to write down consecutively the binomial coefficients for different values of n , as in the following Table :

n	Coefficients.									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
...										

525. The Binomial Theorem may be applied to expand expressions which contain more than two terms.

EXAMPLE. Find the expansion of $(x^2+2x-1)^3$.
Regarding $2x-1$ as a single term,
the expansion $= (x^2+\overline{2x-1})^3$
 $= (x^2)^3 + 3(x^2)^2(2x-1) + 3x^2(2x-1)^2 + (2x-1)^3$
 $= x^6 + 3x^4(2x-1) + 3x^2(4x^2-4x+1) + 8x^3-12x^2+6x-1$
 $= x^6 + 6x^5 + 9x^4 - 4x^3 - 9x^2 + 6x - 1.$

526. The simplest form of the Binomial Theorem is the expansion of $(1+x)^n$. This is obtained from the general formula of Art. 523, by writing 1 for x , and x for a .

$$\begin{aligned}\text{Thus } (1+x)^n &= 1 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n \\ &= 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \dots \\ &\quad + \frac{n(n-1)(n-2) \dots (n-r+1)}{r} x^r + \dots + x^n.\end{aligned}$$

The expansion of a binomial may always be made to depend upon the case in which the first term is unity ;

thus $(x+y)^n = \left\{ x \left(1 + \frac{y}{x} \right) \right\}^n = x^n (1+z)^n$, where $z = \frac{y}{x}$.

EXAMPLE 1. Expand $(1+2x)^4(1-x^2)^3$ as far as the term which involves x^5 .

$$\begin{aligned}(1+2x)^4 &= 1 + 4(2x) + 6(2x)^2 + 4(2x)^3 + (2x)^4 \\ &= 1 + 8x + 24x^2 + 32x^3 + 16x^4, \dots\dots\dots(1)\end{aligned}$$

$$(1-x^2)^3 = 1 - 3x^2 + 3x^4 - x^6. \dots\dots\dots(2)$$

The product of the expressions (1) and (2) may now be obtained by using detached coefficients. In (2) we must write 0 for the coefficients of any missing powers of x , and we may omit all terms which would give rise in the product to powers of x higher than the fifth.

$$\begin{array}{r|l} 1 + 8 + 24 + 32 + 16 & \\ 1 + 0 - 3 + 0 + 3 + 0 & \\ \hline 1 + 8 + 24 + 32 + 16 & \\ \quad - 3 - 24 - 72 - 96 & \\ \quad \quad \quad + 3 + 24 & \\ \hline 1 + 8 + 21 + 8 - 53 - 72 & \end{array}$$

Thus the required result is $1 + 8x + 21x^2 + 8x^3 - 53x^4 - 72x^5$.

EXAMPLE 2. Find the value of $(1+\sqrt{1-x^2})^5 + (1-\sqrt{1-x^2})^5$.

Put y for $\sqrt{1-x^2}$, then we have to find the sum of the expansions of $(1+y)^5$ and $(1-y)^5$. In these the terms are numerically equal, but in the second expansion the second, fourth, and sixth terms are negative and therefore destroy the corresponding terms of the first expansion.

$$\begin{aligned}\text{Hence the required value} &= 2\{1 + {}^5C_2y^2 + {}^5C_4y^4\} \\ &= 2\{1 + 10(1-x^2) + 5(1-x^2)^2\} \\ &= 2\{1 + 10 - 10x^2 + 5 - 10x^2 + 5x^4\} \\ &= 32 - 40x^2 + 10x^4.\end{aligned}$$

[Examples XLI. a. 1-22, page 482, may be taken here.]

527. General Term. In the expansion of $(x+a)^n$, the coefficient of the second term is nC_1 ; of the third term is nC_2 ; of the fourth term is nC_3 ; and so on, *the suffix of C in each term being one less than the number of the term to which it applies*; hence nC_r is the coefficient of the $(r+1)^{\text{th}}$ term. This is called the **general term**, because by giving to r different numerical values any of the coefficients may be found from nC_r ; and by giving to x and a their appropriate indices the value of any assigned term may be obtained.

Thus the $(r+1)^{\text{th}}$ term $= {}^nC_r x^{n-r} a^r$, and may be written

$$\frac{n(n-1)(n-2) \dots (n-r+1)}{r!} x^{n-r} a^r, \quad \text{or} \quad \frac{{}^n_{n-r} r!}{r!} x^{n-r} a^r.$$

In applying the form ${}^nC_r x^{n-r} a^r$ to any particular case, it should be observed that *the index of a is the same as the suffix of C, and that the sum of the indices of x and a is n.*

The symbol T_{r+1} is often used to denote the $(r+1)^{\text{th}}$ term.

EXAMPLE 1. Find the fifth term of $(x+2y^3)^{17}$.

$$\begin{aligned} \text{Here} \quad T_5 &= T_{4+1} = {}^{17}C_4 x^{13} (2y^3)^4 \\ &= \frac{17 \cdot 16 \cdot 15 \cdot 14}{1 \cdot 2 \cdot 3 \cdot 4} \times 16 x^{13} y^{12} \\ &= 38080 x^{13} y^{12}. \end{aligned}$$

EXAMPLE 2. Find the fourteenth term of $(3-a)^{15}$.

$$\begin{aligned} \text{Here} \quad T_{14} &= {}^{15}C_{13} (3)^2 (-a)^{13} \\ &= {}^{15}C_2 \times (-9a^{13}) \\ &= -945a^{13}. \end{aligned}$$

EXAMPLE 3. Find the term containing x^{16} in the expansion of $(x^2-2x)^{10}$.

$$\text{We have} \quad (x^2-2x)^{10} = \left\{ x^2 \left(1 - \frac{2}{x} \right) \right\}^{10} = x^{20} \left(1 - \frac{2}{x} \right)^{10}.$$

Hence the term containing x^{16} will be obtained from the product of x^{20} and the term which contains $\frac{1}{x^4}$ in $\left(1 - \frac{2}{x} \right)^{10}$. This is the term which contains $\left(\frac{2}{x} \right)^7$.

Hence

$$\text{the required term} = x^{20} \times {}^{10}C_7 \left(-\frac{2}{x} \right)^7 = -\frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} \times 2^7 x^{16} = 15360 x^{16}.$$

EXAMPLE 4. Find the coefficient of x^9 in the expansion of $\left(x^2 + \frac{1}{x^3} \right)^{12}$.

$$\text{The general term} = {}^{12}C_r (x^2)^{12-r} \left(\frac{1}{x^3} \right)^r = {}^{12}C_r x^{24-5r}.$$

$$\text{Now if} \quad 24 - 5r = 9, \quad \text{then} \quad r = 3.$$

$$\text{Thus the required coefficient} = {}^{12}C_3 = \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} = 220.$$

EXAMPLES XLI. a

1. As in Art. 519, find the expansions of

(i) $(x+2)(x-1)(x+4)$;

(ii) $(x-3)(x+7)(x-2)(x+4)$;

(iii) $(x-4)(x+5)(x+4)(x-5)$;

(iv) $(a+2b)(a-5b)(a+9b)$.

Verify (iii) by an independent method.

Expand the following binomials:

2. $(x+2)^4$.

3. $(x+3)^5$.

4. $(a-x)^5$.

5. $(a+x)^7$.

6. $(1+b)^8$.

7. $(1-2y)^5$.

8. $\left(1+\frac{x}{2}\right)^6$.

9. $\left(2x+\frac{y}{2}\right)^4$.

10. $\left(2-\frac{x}{2}\right)^6$.

11. $\left(a-\frac{3}{b}\right)^7$.

12. $\left(x-\frac{1}{2x}\right)^5$.

13. $\left(ax+\frac{y}{a}\right)^9$.

Expand and simplify:

14. $(a+b)^5 - (a-b)^5$.

15. $(3-2x)^6 + (3+2x)^6$.

16. $(x-\sqrt{3})^4 + (x+\sqrt{3})^4$.

17. $(\sqrt{2}+1)^6 - (\sqrt{2}-1)^6$.

18. Shew that $(x-\sqrt{1-x^2})^4 + (x+\sqrt{1-x^2})^4 = 2 + 8x^2 - 8x^4$.

Expand the following trinomials:

19. $(x^2-x-2)^3$.

20. $(1+x+x^2)^4$.

21. $(1-2a+3a^2)^3$.

22. Expand (i) $(a+x)^6(a-x)^2$;

(ii) $(1-x)^5(1+2x)^3$.

Write down and simplify:

23. The 4th term of $(1+2x)^7$.

24. The 6th term of $(2-y)^8$.

25. The 5th term of $(a-5b)^8$.

26. The 15th term of $(2x-1)^{17}$.

27. The 7th term of $\left(1-\frac{1}{x}\right)^{10}$.

28. The 6th term of $\left(3x+\frac{a}{2}\right)^9$.

29. The middle term of (i) $\left(x^2+\frac{1}{x}\right)^6$;

(ii) $\left(3a-\frac{1}{2a}\right)^8$.

30. The two middle terms of $\left(x-\frac{1}{x}\right)^9$.

31. The 6th term of $\left(\frac{2a}{3}-\frac{3}{2a}\right)^{10}$.

32. The 23rd term of $\left(x^2+\frac{b}{x}\right)^{25}$.

33. Find the coefficient of x^{16} in the expansion of $(x^2-2x)^{10}$.

34. Find the coefficient of x in the expansion of $\left(x^2-\frac{a}{2x}\right)^{14}$.

35. Find the coefficients of x^4 and x^{-1} in $\left(x^3-\frac{1}{x^2}\right)^8$.

36. Find the term independent of x in $\left(2x^2+\frac{1}{x}\right)^{12}$.

37. Shew that the coefficient of the middle term of $(1+x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of $(1+x)^{2n-1}$.

38. If a_r denotes the coefficient of x^r in the expansion of $(1-x)^{2m-1}$, prove that $a_{r-1} + a_{2m-r} = 0$.

528. In the expansion of $(1+x)^n$ the coefficients of terms equidistant from the beginning and end are equal.

The coefficient of the $(r+1)^{\text{th}}$ term from the beginning is nC_r .

The $(r+1)^{\text{th}}$ term from the end has $(n+1)-(r+1)$, or $n-r$ terms before it; therefore counting from the beginning it is the $(n-r+1)^{\text{th}}$ term, and its coefficient is ${}^nC_{n-r}$, which has been shewn to be equal to nC_r . [Art. 508.] Hence the proposition follows.

529. To find the greatest term in the expansion of $(1+x)^n$.

Here
$$T_{r+1} = \frac{n(n-1)(n-2) \dots (n-r+2)(n-r+1)}{1 \cdot 2 \cdot 3 \dots r} x^r,$$

$$T_r = \frac{n(n-1)(n-2) \dots (n-r+2)}{1 \cdot 2 \cdot 3 \dots (r-1)} x^{r-1}.$$

$$\therefore T_{r+1} = T_r \times \frac{n-r+1}{r} x.$$

Hence the $(r+1)^{\text{th}}$ term is obtained by multiplying the r^{th} term by $\frac{n-r+1}{r} x$; that is, by $\left(\frac{n+1}{r} - 1\right) x$.

The factor $\frac{n+1}{r} - 1$ decreases as r increases; hence the $(r+1)^{\text{th}}$ term is not always greater than the r^{th} term, but only until $\left(\frac{n+1}{r} - 1\right) x$ becomes equal to 1, or less than 1.

Now $\left(\frac{n+1}{r} - 1\right) x > 1$, so long as $\frac{n+1}{r} - 1 > \frac{1}{x}$;

that is, $\frac{n+1}{r} > \frac{1}{x} + 1$, or $\frac{(n+1)x}{1+x} > r$(1)

If $\frac{(n+1)x}{1+x}$ is an integer, denote it by p ; then if $r=p$ the multiplying factor becomes equal to 1, and the $(p+1)^{\text{th}}$ term is equal to the p^{th} ; and these are greater than any other term.

If $\frac{(n+1)x}{1+x}$ is not an integer, denote its integral part by q ; then the greatest value of r consistent with (1) is q ; hence the $(q+1)^{\text{th}}$ term is the greatest.

Since we are only concerned with the *numerically* greatest term, the investigation will be the same for $(1-x)^n$; therefore in any numerical example it is unnecessary to consider the sign of the second term of the binomial.

NOTE. To find the greatest *coefficient* in $(1+x)^n$ we have only to find, as in Art. 512, the value of r which makes nC_r greatest.

530. If a binomial is given in the form $(x+a)^n$,

we have
$$(x+a)^n = x^n \left(1 + \frac{a}{x}\right)^n;$$

therefore, since x^n multiplies every term in $\left(1 + \frac{a}{x}\right)^n$, it will be sufficient to find which is the greatest term in this latter expansion.

EXAMPLE. Find the greatest term in the expansion of $(x-4a)^8$, when $x = \frac{1}{2}$ and $a = \frac{1}{3}$.

Since $(x-4a)^8 = x^8 \left(1 - \frac{4a}{x}\right)^8$, it will be sufficient to find which is the greatest term of $\left(1 - \frac{4a}{x}\right)^8$.

Here
$$T_{r+1} = T_r \times \frac{8-r+1}{r} \cdot \frac{4a}{x}, \text{ numerically,}$$

$$= T_r \times \frac{9-r}{r} \cdot \frac{8}{3}.$$

Hence
$$T_{r+1} > T_r \text{ so long as } \frac{(9-r)8}{3r} > 1;$$

that is,
$$72 - 8r > 3r, \text{ or } 72 > 11r.$$

The greatest value of r consistent with this is 6; hence the greatest term is the 7th.

The 7th term of $(x-4a)^8 = {}^8C_6 x^2 (-4a)^6 = {}^8C_2 \times \frac{1}{2^2} \times \left(\frac{4}{3}\right)^6 = \frac{28672}{729}.$

Properties of the Binomial Coefficients

531. To find the sum of the coefficients in the expansion of $(1+x)^n$.

For the sake of brevity we shall now express the coefficients by the symbols $c_0, c_1, c_2, c_3, \dots, c_n$, so that

$$(1+x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots + c_nx^n.$$

Put $x=1$; then $2^n = c_0 + c_1 + c_2 + c_3 + \dots + c_n$
 $= \text{the sum of the coefficients.}$

NOTE. Since $c_0=1$, we have $c_1 + c_2 + c_3 + \dots + c_n = 2^n - 1$. which is the result proved in Art. 510.

532. In the expansion of $(1+x)^n$ the sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms.

We have
$$(1+x)^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots + c_nx^n.$$

Put $x=-1$, then $0 = c_0 - c_1 + c_2 - c_3 + c_4 - c_5 + \dots$

$$\therefore c_0 + c_2 + c_4 + \dots = c_1 + c_3 + c_5 + \dots$$

By Art. 531, each of these equal expressions $= \frac{1}{2} \cdot 2^n = 2^{n-1}.$

533. To find the sum of the squares of the coefficients in the expansion of $(1+x)^n$.

We have $(1+x)^n = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$

and $(x+1)^n = c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \dots + c_n$.

If we multiply together the two series on the right, the coefficient of x^n is

$$c_0^2 + c_1^2 + c_2^2 + \dots + c_n^2.$$

Therefore this expression must be equal to the coefficient of x^n in the product $(1+x)^n(x+1)^n$, or $(1+x)^{2n}$. [See Art. 474.]

$$\therefore c_0^2 + c_1^2 + c_2^2 + \dots + c_n^2 = \frac{\begin{smallmatrix} 2n \\ n \end{smallmatrix}}{\begin{smallmatrix} n \\ n \end{smallmatrix}}.$$

NOTE. The following method shews a device often useful.

$$(1+x)^n = c_0 + c_1x + c_2x^2 + \dots + c_nx^n.$$

Writing $\frac{1}{x}$ for x , we have

$$\left(1 + \frac{1}{x}\right)^n, \text{ or } \frac{1}{x^n}(1+x)^n = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots + \frac{c_n}{x^n}.$$

$\therefore$ by multiplying these two results together,

$$\frac{1}{x^n}(1+x)^{2n} = (c_0^2 + c_1^2 + c_2^2 + \dots + c_n^2) + \text{other terms all containing } x.$$

$\therefore c_0^2 + c_1^2 + c_2^2 + \dots + c_n^2 =$ the coefficient of the term independent of x in the expansion of $\frac{1}{x^n}(1+x)^{2n}$

$$= \text{the coefficient of } x^n \text{ in } (1+x)^{2n} = \frac{\begin{smallmatrix} 2n \\ n \end{smallmatrix}}{\begin{smallmatrix} n \\ n \end{smallmatrix}}.$$

EXAMPLE. If $(1+x)^n = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$, prove that

$$(i) \ c_0 + 2c_1 + 3c_2 + 4c_3 + \dots + (n+1)c_n = 2^n + n \cdot 2^{n-1};$$

$$(ii) \ c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2 + \frac{1}{4}c_3 + \dots + \frac{1}{n+1} \cdot c_n = \frac{2^{n+1} - 1}{n+1}.$$

$$(i) \text{ The series } = (c_0 + c_1 + c_2 + \dots + c_n) + (c_1 + 2c_2 + 3c_3 + \dots + nc_n)$$

$$= 2^n + n \left\{ 1 + (n-1) + \frac{(n-1)(n-2)}{1 \cdot 2} + \dots + 1 \right\}$$

$$= 2^n + n(1+1)^{n-1} = 2^n + n \cdot 2^{n-1}.$$

$$(ii) \text{ Here } (n+1) \left\{ c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2 + \frac{1}{4}c_3 + \dots + \frac{1}{n+1} \cdot c_n \right\}$$

$$= (n+1) + \frac{(n+1)n}{2} + \frac{(n+1)n(n-1)}{\begin{smallmatrix} 3 \\ 3 \end{smallmatrix}} + \dots \text{ to } n+1 \text{ terms}$$

$$= \left\{ 1 + (n+1) + \frac{(n+1)n}{2} + \frac{(n+1)n(n-1)}{\begin{smallmatrix} 3 \\ 3 \end{smallmatrix}} + \dots \text{ to } n+2 \text{ terms} \right\} - 1$$

$$= (1+1)^{n+1} - 1 = 2^{n+1} - 1.$$

Dividing by $n+1$ we obtain the required result.

EXAMPLES XLI. b

In the following expansions find which is the greatest term :

1. $(1-x)^{25}$ when $x=3$.
2. $(x+y)^{15}$ when $x=5, y=2$.
3. $(a+7b)^n$ when $n=19, a=14, b=3$.
4. $(2x-3a)^n$ when $n=13, a=4, x=9$.
5. $\left(5a - \frac{b}{5}\right)^n$ when $n=16$, and $b=10a$.

In the following expansions find the value of the greatest term :

6. $(1+2x)^9$ when $x=\frac{1}{3}$.
 7. $(2+3x)^8$ when $x=\frac{1}{2}$.
 8. Shew that in the expansion of $\left(\frac{1}{5} + \frac{5x}{16}\right)^{12}$ there are two greatest terms, each equal to $\frac{99}{8^4 \times 5^7}$, when $x=\frac{2}{5}$.
 9. Find the numerically greatest *coefficient* in the expansion of
 - (i) $\left(1 + \frac{2x}{3}\right)^{12}$;
 - (ii) $(3-5x)^8$.
 10. In the expansion of $(1+x)^{15}$ the coefficients of the $(r-1)^{\text{th}}$ and $(2r+3)^{\text{th}}$ terms are equal ; find r .
 11. Find n when the coefficients of the 9th and 15th terms of $(1+x)^n$ are equal.
 12. Find the sum of the coefficients of $(x+y)^{12}$.
 13. Find the sum of the coefficients of $(2x+3y)^5$.
- If $(1+x)^n = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$, prove that
14. $c_1 + 2c_2 + 3c_3 + \dots + nc_n = n \cdot 2^{n-1}$.
 15. $c_2 + 2c_3 + 3c_4 + \dots + (n-1)c_n = 1 + (n-2)2^{n-1}$.
 16. $c_0 + c_1x + 2c_2x^2 + \dots + nc_nx^n = 1 + nx(1+x)^{n-1}$.
 17. $c_1 - 2c_2 + 3c_3 - \dots + (-1)^{n-1}nc_n = 0$.
 18. $c_0 - \frac{c_1}{2} + \frac{c_2}{3} - \dots + (-1)^n \cdot \frac{c_n}{n+1} = \frac{1}{n+1}$.
 19. $\frac{c_1}{c_0} + \frac{2c_2}{c_1} + \frac{3c_3}{c_2} + \dots + \frac{nc_n}{c_{n-1}} = \frac{n(n+1)}{2}$.
 20. $2c_0 + \frac{2^2c_1}{2} + \frac{2^3c_2}{3} + \dots + \frac{2^{n+1}c_n}{n+1} = \frac{3^{n+1}-1}{n+1}$.
 21. $c_0c_1 + c_1c_2 + c_2c_3 + \dots + c_{n-1}c_n = \frac{2n}{(n+1)(n-1)}$.

(Miscellaneous)

22. Find the coefficient of x^3 in the expansion of $\left(32x - \frac{1}{2}\right)^{20}$.
23. Expand $(3 - 2x)^6$.
24. Find the term containing x^7 in the expansion of $\left(x^2 - \frac{1}{x}\right)^{20}$.
25. Find the first four terms in the expansion of $\left(1 + \frac{1}{x}\right)^x$, and write down the general term in the neatest form.
If $x = 100$, find the sum of the first three terms.
26. Find the r^{th} term from the beginning and the r^{th} term from the end of $(a + 2x)^n$.
27. Find the coefficient of x^4 in the expansion of $(1 - x - x^2)^{10}$.
28. Shew that the term independent of x in the expansion of $\left(2x - \frac{3}{x^2}\right)^9$ is $-2^8 \times 3^4 \times 7$.
29. Find the value of $(1.012)^5$ to three places of decimals.
30. Find the coefficient of x^4 in the expansion of $(1 + x + 2x^2)^4$.
31. If $(1 + x)^3 = 1 + a_1x + a_2x^2 + \dots$
and $(1 + x)^5 = 1 + b_1x + b_2x^2 + \dots$,
find the value of $1 + a_1b_1 + a_2b_2 + a_3b_3$.
32. Find the term independent of x in the expansion of $\left(2x^2 - \frac{a}{2x^3}\right)^{10}$.
33. Shew that $(1 - x^2)^n$ may be put in the form

$$(1 + x)^{2n} - 2nx(1 + x)^{2n-1} + \frac{2n(2n-2)}{1 \cdot 2} x^2(1 + x)^{2n-2} - \dots$$
34. Prove that the coefficient of x^n in $(1 + x)^{2n}$ is equal to twice the coefficient of x^n in $(1 + x)^{2n-1}$.
35. Find the coefficient of x^r in the expansion of $\left(x^2 + \frac{1}{x^3}\right)^n$. Shew that $2n - r$ and $3n + r$ must each be a multiple of 5.
36. If A stands for the sum of the odd terms and B for the sum of the even terms in the expansion of $(x + a)^n$, prove that

$$A^2 - B^2 = (x^2 - a^2)^n.$$
37. Shew that the middle term in the expansion of $(1 + x)^{2n}$ can be expressed in the form $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{n} 2^n x^n$.
38. Shew that the difference between the coefficients of x^{r+1} and x^r in the expansion of $(1 + x)^{n+1}$ is equal to the difference between the coefficients of x^{r+1} and x^{r-1} in the expansion of $(1 + x)^n$.

Binomial Theorem for Negative and Fractional Indices

534. We have now to consider whether the series

$$1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \dots + \frac{n(n-1)(n-2) \dots (n-r+1)}{1 \cdot 2 \cdot 3 \dots r} x^r + \dots$$

is a true equivalent of $(1+x)^n$ when n is negative or fractional. When n is a positive integer the series ends with the $(n+1)^{\text{th}}$ term (Art. 522), but if n is negative or fractional none of the factors in the numerator of the general term can ever vanish; thus the series becomes infinite since none of its terms can become zero.

By actual division we can shew that

$$(1-x)^{-2} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots,$$

and as the process of division is unending, the number of terms in the series is unlimited.

Now let us assume for the moment that the Binomial Theorem is true for a negative index; then by putting $n = -2$ and writing $-x$ for x in the formula for $(1+x)^n$, we have

$$\begin{aligned} (1-x)^{-2} &= 1 + (-2)(-x) + \frac{(-2)(-3)}{1 \cdot 2} (-x)^2 + \frac{(-2)(-3)(-4)}{1 \cdot 2 \cdot 3} (-x)^3 + \dots \\ &= 1 + 2x + 3x^2 + 4x^3 + \dots \end{aligned}$$

Thus the Binomial Theorem appears to hold good in this case. But if we were to make trial of a particular value of x , such as $x=2$, we should have

$$(-1)^{-2} = 1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \dots$$

This contradictory result is sufficient to shew that we cannot take the series

$$1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \dots$$

as the true arithmetical equivalent of $(1+x)^n$ in *all* cases.

Again, by putting $n = -1$, and writing $-x$ for x in the Binomial Formula, we should get

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$$

The series on the right is a G.P. and the sum of the first r terms

$$= \frac{1-x^r}{1-x} = \frac{1}{1-x} - \frac{x^r}{1-x}.$$

If x is numerically less than 1, we can make x^r as small as we please by taking r large enough. Thus the sum to infinity

$$= \frac{1}{1-x} = (1-x)^{-1}.$$

From this we may infer that the expansion of $(1-x)^{-1}$ by the Binomial Theorem is true when x is numerically less than 1.

535. If the sum of n terms of a series tends to a finite limit, which it cannot exceed, when n is made infinitely great, the series is said to be **convergent**, and the finite limit to which it converges is called the **sum to infinity**.

Thus the series $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$ cannot exceed the limit 2 (Art. 330), and is therefore convergent.

If by increasing n sufficiently, the sum of n terms of a series can be made to exceed any quantity, however large, the series is said to be **divergent**.

Thus the series $1 + 2 + 4 + 8 + \dots$ can be made greater than any quantity that can be named, by taking a sufficient number of terms, and is therefore divergent.

Again, the sum of the first n terms of the series

$$\begin{aligned} a + ar + ar^2 + ar^3 + \dots &= \frac{a(1 - r^n)}{1 - r} \\ &= \frac{a}{1 - r} - \frac{ar^n}{1 - r}. \end{aligned}$$

If r is numerically less than 1, the sum approaches to the finite limit $\frac{a}{1 - r}$ when n is infinitely great, and the series is convergent.

If r is numerically greater than 1, the sum of the first n terms is $\frac{ar^n}{r - 1} - \frac{a}{r - 1}$, which can be made greater than any finite quantity by taking n large enough; thus the series is divergent.

The consideration of the convergency or divergency of series is beyond the elementary scope of this book. The more advanced reader will find the subject treated in Hall and Knight's *Higher Algebra*, Chap. XXI. For a much fuller discussion he may consult Chrystal's *Algebra*, Part II., Chap. XXVI. It will be sufficient here to say generally that divergent series are practically of no importance in algebraical work, but that convergent series may be introduced into mathematical reasoning as freely as any other functions to which the laws of Algebra are applicable.

It can be proved that in the Binomial Formula

$$\begin{aligned} (1 + x)^n &= 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \dots \\ &\quad + \frac{n(n-1)(n-2) \dots (n-r+1)}{r} x^r + \dots, \end{aligned}$$

the series on the right is convergent for all values of n , provided that x is numerically less than 1, and that with this restriction the expansion is a true arithmetical equivalent of $(1 + x)^n$.

536. Henceforth we shall assume that the formula

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \dots$$

is true for any value of n so long as x is numerically less than 1. We may again remark that the series on the right is infinite when n is negative or fractional, but in any particular case we may write down as many terms as we please, or we may find the coefficient of any assigned term.

If we have to expand $(a+x)^n$ we may write the expression

$$a^n \left(1 + \frac{x}{a}\right)^n, \quad \text{or} \quad x^n \left(1 + \frac{a}{x}\right)^n;$$

we must then use the first or second of these forms according as x is less or greater than a .

In the following examples we shall assume that the values of the symbols are such as to make the expansion possible.

EXAMPLE 1. Expand $(1+x)^{-3}$ to four terms.

$$\begin{aligned} (1+x)^{-3} &= 1 + (-3)x + \frac{(-3)(-3-1)}{1 \cdot 2} x^2 + \frac{(-3)(-3-1)(-3-2)}{1 \cdot 2 \cdot 3} x^3 + \dots \\ &= 1 - 3x + \frac{3 \cdot 4}{1 \cdot 2} x^2 - \frac{3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3} x^3 + \dots \\ &= 1 - 3x + 6x^2 - 10x^3 + \dots \end{aligned}$$

EXAMPLE 2. Expand $(4+3x)^{\frac{3}{2}}$ to four terms.

$$\begin{aligned} (4+3x)^{\frac{3}{2}} &= 4^{\frac{3}{2}} \left(1 + \frac{3x}{4}\right)^{\frac{3}{2}} = 8 \left(1 + \frac{3x}{4}\right)^{\frac{3}{2}} \\ &= 8 \left[1 + \frac{3}{2} \cdot \frac{3x}{4} + \frac{\frac{3}{2}(\frac{3}{2}-1)}{1 \cdot 2} \left(\frac{3x}{4}\right)^2 + \frac{\frac{3}{2}(\frac{3}{2}-1)(\frac{3}{2}-2)}{1 \cdot 2 \cdot 3} \left(\frac{3x}{4}\right)^3 + \dots \right] \\ &= 8 \left[1 + \frac{3}{2} \cdot \frac{3x}{4} + \frac{3}{8} \cdot \frac{9x^2}{16} - \frac{1}{16} \cdot \frac{27x^3}{64} + \dots \right] \\ &= 8 + 9x + \frac{27}{16} x^2 - \frac{27}{128} x^3 + \dots \end{aligned}$$

EXAMPLE 3. Find the general term in the expansion of $(1+x)^{\frac{1}{2}}$.

$$\begin{aligned} T_{r+1} &= \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-r+1)}{\underbrace{1 \cdot 2 \cdot 3 \dots r}_r} x^r \\ &= \frac{1(-1)(-3)(-5)\dots(-2r+3)}{2^r \underbrace{1 \cdot 2 \cdot 3 \dots r}_r} x^r. \end{aligned}$$

The number of factors in the numerator is r , and $r-1$ of these are negative; therefore, by taking -1 out of each of these negative factors, we may write the above expression

$$(-1)^{r-1} \frac{1 \cdot 3 \cdot 5 \dots (2r-3)}{2^r \underbrace{1 \cdot 2 \cdot 3 \dots r}_r} x^r.$$

EXAMPLE 4. Find the general term in the expansion of $(1-x)^{-3}$.

$$\begin{aligned} T_{r+1} &= \frac{(-3)(-4)(-5) \dots (-3-r+1)}{\underbrace{\hspace{1.5cm}}^r} (-x)^r \\ &= (-1)^r \frac{3 \cdot 4 \cdot 5 \dots (r+2)}{\underbrace{\hspace{1.5cm}}^r} (-1)^r x^r \\ &= (-1)^{2r} \frac{3 \cdot 4 \cdot 5 \dots (r+2)}{1 \cdot 2 \cdot 3 \dots r} x^r \\ &= \frac{(r+1)(r+2)}{1 \cdot 2} x^r, \end{aligned}$$

by removing like factors from the numerator and denominator.

[Examples XLI. c. 1-18, page 492, may be taken here.]

537. To find in its simplest form the general term in the expansion of $(1-x)^{-n}$.

$$\begin{aligned} T_{r+1} &= \frac{(-n)(-n-1)(-n-2) \dots (-n-r+1)}{\underbrace{\hspace{1.5cm}}^r} (-x)^r \\ &= (-1)^r \frac{n(n+1)(n+2) \dots (n+r-1)}{\underbrace{\hspace{1.5cm}}^r} (-1)^r x^r \\ &= (-1)^{2r} \frac{n(n+1)(n+2) \dots (n+r-1)}{\underbrace{\hspace{1.5cm}}^r} x^r \\ &= \frac{n(n+1)(n+2) \dots (n+r-1)}{\underbrace{\hspace{1.5cm}}^r} x^r. \end{aligned}$$

From this it appears that every term in the expansion of $(1-x)^{-n}$ is positive, and this form may be conveniently used in all cases when the index is negative.

EXAMPLE. Find the general term of $\frac{1}{\sqrt[3]{1-3x}}$.

We require the $(r+1)^{\text{th}}$ term of $(1-3x)^{-\frac{1}{3}}$.

$$\begin{aligned} T_{r+1} &= \frac{\frac{1}{3}(\frac{1}{3}+1)(\frac{1}{3}+2) \dots (\frac{1}{3}+r-1)}{\underbrace{\hspace{1.5cm}}^r} (3x)^r \\ &= \frac{1 \cdot 4 \cdot 7 \dots (3r-2)}{3^r \underbrace{\hspace{1.5cm}}^r} 3^r x^r \\ &= \frac{1 \cdot 4 \cdot 7 \dots (3r-2)}{\underbrace{\hspace{1.5cm}}^r} x^r. \end{aligned}$$

If the given expression had been $(1+3x)^{-\frac{1}{3}}$ we should have used the same formula for the general term, replacing $3x$ by $-3x$.

538. The following results should be verified and noted.

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots + x^r + \dots,$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots + (r+1)x^r + \dots,$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots + \frac{(r+1)(r+2)}{1 \cdot 2} x^r + \dots$$

EXAMPLES XLI. c

Expand to 4 terms the following expressions :

1. $(1+x)^{\frac{1}{3}}$.
2. $(1+x)^{\frac{3}{4}}$.
3. $(1-x)^{\frac{2}{5}}$.
4. $(1+3x)^{-2}$.
5. $(1+x^2)^{-3}$.
6. $(1+3x)^{-4}$.
7. $(2+x)^{-3}$.
8. $(1+2x)^{-\frac{1}{2}}$.
9. $(a-2x)^{-\frac{3}{2}}$.
10. $(1-x)^{\frac{5}{2}}$.
11. $(9+2x)^{\frac{1}{2}}$.
12. $(8+12a)^{\frac{2}{3}}$.

Write down and simplify :

13. The 5th term and the 10th term of $(1+x)^{-\frac{3}{2}}$.
14. The 3rd term and the 11th term of $(1+2x)^{\frac{11}{2}}$.
15. The 4th term and the $(r+1)$ th term of $(1+x)^{-2}$.
16. The 7th term and the $(r+1)$ th term of $(1-x)^{\frac{1}{2}}$.
17. The $(r+1)$ th term of $(a-bx)^{-1}$, and of $(1-nx)^{\frac{1}{n}}$.
18. The $(r+1)$ th term of $(1+x)^{\frac{5}{2}}$, and of $(1+2x)^{-\frac{3}{2}}$.

Find the general term in each of the following expansions :

19. $(1-x)^{-5}$.
20. $(1+x^2)^{-3}$.
21. $(1+x)^{-\frac{1}{2}}$.
22. $(1+x)^{-\frac{2}{3}}$.
23. $(2-x)^{-10}$.
24. $(1+x)^{-\frac{p}{q}}$.
25. $\frac{1}{\sqrt{1+2x}}$.
26. $\frac{1}{\sqrt[3]{(1-3x)^2}}$.
27. Find the general term in the expansion of $(1-x)^{\frac{2}{3}}$, and shew that all the terms after the first are negative.
28. Find the first negative term in each of the following expansions :
 (i) $(1+2x)^{\frac{5}{2}}$; (ii) $(1+x)^{\frac{4}{3}}$; (iii) $(1+3x)^{\frac{11}{3}}$.

539. To find the greatest term in the expansion of $(1+x)^n$ when n is unrestricted in value, we may proceed exactly as in Art. 530. When the index is negative, we may conveniently use the form for the general term given in Art. 537.

EXAMPLE. Find which is the greatest term in the expansion of $(3+7x)^{-n}$, when $x = \frac{3}{8}$, $n = \frac{11}{4}$.

Since $(3+7x)^{-\frac{11}{4}} = 3^{-\frac{11}{4}} \left(1 + \frac{7x}{3}\right)^{-\frac{11}{4}}$, it will be sufficient to find the greatest term of $\left(1 + \frac{7x}{3}\right)^{-\frac{11}{4}}$.

Here

$$T_{r+1} = T_r \times \frac{\frac{11}{4} + r - 1}{r} \cdot \frac{7x}{3}, \text{ numerically,}$$

$$= T_r \times \frac{7+4r}{4r} \cdot \frac{7}{8}.$$

Hence

$$T_{r+1} > T_r, \text{ so long as } \frac{(7+4r)7}{32r} > 1;$$

that is,

$$49 + 28r > 32r, \text{ or } 49 > 4r.$$

The greatest value of r consistent with this is 12; hence the greatest term is the 13th.

540. The methods used in the following examples deserve attention.

EXAMPLE 1. Find the coefficient of x^r in the expansion of $\frac{2+x}{(1+x)^3}$.

The expression $= (2+x)(1+x)^{-3}$

$= (2+x)(1+p_1x+p_2x^2+\dots+p_rx^r+\dots)$, suppose.

$\therefore$ the coefficient of $x^r = p_{r-1} + 2p_r$.

Now $p_r = (-1)^r \frac{(r+1)(r+2)}{2}$. [Art. 538]

$\therefore$ the required coefficient $= (-1)^{r-1} \frac{r(r+1)}{2} + (-1)^r (r+1)(r+2)$

$$= (-1)^r (r+1) \left\{ r+2 - \frac{r}{2} \right\}$$

$$= (-1)^r \frac{(r+1)(r+4)}{2}.$$

EXAMPLE 2. Find the value of the infinite series

$$1 - \frac{1}{2} \cdot \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{2^2} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{2^3} + \dots$$

The series $= 1 - \frac{1}{2} \cdot \frac{1}{2} + \frac{\frac{1}{2} \cdot \frac{3}{2}}{1 \cdot 2} \cdot \frac{1}{2^2} - \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2^3} + \dots$

$$= 1 - \frac{1}{2} \cdot \frac{1}{2} + \frac{\frac{1}{2}(\frac{1}{2}+1)}{1 \cdot 2} \cdot \frac{1}{2^2} - \frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2^3} + \dots$$

$$= \left(1 + \frac{1}{2}\right)^{-\frac{1}{2}} = \left(\frac{3}{2}\right)^{-\frac{1}{2}} = \left(\frac{2}{3}\right)^{\frac{1}{2}} = \sqrt{\frac{2}{3}}.$$

EXAMPLES XLI. d

Find which is the greatest term in the following expansions :

1. $(1+x)^{-5}$ when $x = \frac{5}{8}$.

2. $(1+x)^{\frac{11}{2}}$ when $x = \frac{2}{5}$.

3. $(1-x)^{-20}$ when $x = \frac{2}{3}$.

4. $(9+3x)^{\frac{3}{2}}$ when $x = 2$.

5. $(6+5x)^{\frac{31}{5}}$ when $x = \frac{4}{5}$.

6. $\left(\frac{2x}{9} - \frac{x^2}{4}\right)^{-\frac{3}{x}}$ when $x = \frac{1}{12}$.

7. Find the numerically greatest coefficient in the expansion of

(i) $\left(1 + \frac{5x}{6}\right)^{\frac{3}{2}}$; (ii) $(3-4x)^{\frac{11}{4}}$.

8. Find the coefficient of x^{100} in the following expansions :

(i) $\frac{(1+x)^2}{1-x}$; (ii) $\frac{1-x}{(1+x)^2}$; (iii) $\frac{x}{(1+x)^3}$.

9. Find the coefficient of x^r in the expansions of

$$(i) \frac{1+x+x^2}{(1-x)^2}; \quad (ii) \frac{1-x+x^2}{(1-x)^3}; \quad (iii) \frac{1+3x}{(1+x)^2}.$$

Verify each case when $r=3$.

10. Shew that

$$(4+2x+3x^2)/(1-x)^2 = 4 + 10x + 19x^2 + 28x^3 + \dots + (9r+1)x^r + \dots$$

11. Find the coefficient of x^r in the expansion of $(2-3x)/(1+x)^4$.

Find the sum to infinity of the following series :

$$12. \quad 1 - \frac{1}{6} + \frac{1 \cdot 3}{6 \cdot 12} - \frac{1 \cdot 3 \cdot 5}{6 \cdot 12 \cdot 18} + \dots \quad 13. \quad 1 + \frac{1}{6} + \frac{1 \cdot 4}{3 \cdot 6} \cdot \frac{1}{4} + \frac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9} \cdot \frac{1}{8} + \dots$$

Shew that

$$14. \quad 1 - \frac{1}{4} \cdot \frac{1}{2} + \frac{1 \cdot 3}{4 \cdot 8} \cdot \frac{1}{2^2} - \frac{1 \cdot 3 \cdot 5}{4 \cdot 8 \cdot 12} \cdot \frac{1}{2^3} + \dots \text{ to infinity} = \frac{2}{5}\sqrt{5}.$$

$$15. \quad 1 + \frac{3}{4} + \frac{3 \cdot 5}{4 \cdot 8} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12} + \dots \text{ to infinity} = \sqrt{8}.$$

$$16. \quad 1 + \frac{1}{3} \cdot \frac{1}{4} + \frac{1 \cdot 4}{3 \cdot 6} \cdot \frac{1}{4^2} + \frac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9} \cdot \frac{1}{4^3} + \dots \text{ to infinity} = \sqrt[3]{\frac{4}{3}}.$$

$$17. \quad 1 + \frac{7}{3 \cdot 8} + \frac{1 \cdot 4 \cdot 7^2}{3^2 \cdot 8^2} \cdot \frac{1}{2} + \frac{1 \cdot 4 \cdot 7 \cdot 7^3}{3^3 \cdot 8^3} \cdot \frac{1}{3} + \dots \text{ to infinity} = 2.$$

18. Find the sum of the infinite series whose $(n+1)^{\text{th}}$ term is

$$\frac{(2n+2)(2n+4)(2n+6)(2n+8)}{3 \cdot 6 \cdot 9 \cdot 12} \cdot \frac{1}{3^n}.$$

19. Prove that

$$\left(\frac{1+2x}{1+x}\right)^n = 1 + n \cdot \left(\frac{x}{1+2x}\right) + \frac{n(n+1)}{1 \cdot 2} \cdot \left(\frac{x}{1+2x}\right)^2 + \dots$$

$$\left[\text{Note that } \left(\frac{1+2x}{1+x}\right)^n = \left(\frac{1+x}{1+2x}\right)^{-n} = \left(\frac{1+2x-x}{1+2x}\right)^{-n} \right]$$

20. Prove that

$$\left(\frac{1+x}{1-x}\right)^{\frac{n}{2}} = 1 + n \cdot \left(\frac{x}{1+x}\right) + \frac{n(n+2)}{1 \cdot 2} \cdot \left(\frac{x}{1+x}\right)^2 + \frac{n(n+2)(n+4)}{1 \cdot 2 \cdot 3} \cdot \left(\frac{x}{1+x}\right)^3 + \dots$$

$$21. \quad \text{Prove that } 1 + \frac{3n}{4} + \frac{3n(3n+3)}{1 \cdot 2} \cdot \frac{1}{4^2} + \frac{3n(3n+3)(3n+6)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{4^3} + \dots$$

$$= 3^n \left\{ 1 + \frac{n}{4} + \frac{n(n+1)}{1 \cdot 2} \cdot \frac{1}{4^2} + \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} \cdot \frac{1}{4^3} + \dots \right\}$$

$$= 1 + 3n + \frac{n(n-1)}{1 \cdot 2} \cdot 3^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \cdot 3^3 + \dots$$

22. Find the coefficient of x^r in

$$(i) \sqrt{1+x+x^2+x^3+\dots} \text{ to inf.};$$

$$(ii) (1+2x+3x^2+4x^3+\dots \text{ to inf.})^2;$$

$$(iii) (1-3x+6x^2-10x^3+\dots \text{ to inf.})^{\frac{1}{3}};$$

$$(iv) (1-2x+3x^2-4x^3+\dots \text{ to inf.})^{-n}.$$

Application of the Binomial Theorem to Approximations

541. If x is a small quantity, so that the successive powers x^2 , x^3 , x^4 , are rapidly diminishing, a few terms of the series

$$1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 \dots\dots$$

may be taken as an approximation to the value of $(1+x)^n$. The number of terms to be taken in each case will depend on the value of x , and on the degree of accuracy required.

EXAMPLE 1. Find the value of $(1.04)^5$ by the Binomial Theorem.

$$\begin{aligned} (1.04)^5 &= \left(1 + \frac{4}{100}\right)^5 = \left(1 + \frac{4}{10^2}\right)^5 \\ &= 1 + 5 \cdot \frac{4}{10^2} + 10 \cdot \frac{4^2}{10^4} + 10 \cdot \frac{4^3}{10^6} + 5 \cdot \frac{4^4}{10^8} + \frac{4^5}{10^{10}} \\ &= 1 + \frac{2}{10} + \frac{16}{10^3} + \frac{64}{10^5} + \frac{128}{10^7} + \frac{1024}{10^{10}} \\ &= 1 + 0.2 + 0.016 + 0.00064 + 0.0000128 + 0.0000001024. \end{aligned}$$

Here the successive terms diminish very rapidly, and if we only want an approximate result, it is obvious that all the terms need not be calculated. For example, to obtain a result true to three decimal places we may omit the last two terms, since the fourth begins with 4 ciphers.

Thus $(1.04)^5 = 1.2166 \dots\dots = 1.217$, correct to three decimal places.

EXAMPLE 2. Find the fourth root of 621 to four decimal places.

The complete fourth power nearest to 621 is 625, or 5^4 ; hence we proceed as follows :

$$\begin{aligned} \sqrt[4]{621} &= (5^4 - 4)^{\frac{1}{4}} = 5 \left(1 - \frac{4}{5^4}\right)^{\frac{1}{4}} \\ &= 5 \left\{ 1 - \frac{1}{4} \cdot \frac{4}{5^4} - \frac{3}{32} \cdot \frac{4^2}{5^8} - \frac{7}{128} \cdot \frac{4^3}{5^{12}} - \dots\dots \right\} \\ &= 5 - \frac{1}{5^3} - \frac{3}{2} \cdot \frac{1}{5^7} - \frac{7}{2} \cdot \frac{1}{5^{11}} - \dots\dots \\ &= 5 - \frac{2^3}{10^3} - 3 \cdot \frac{2^6}{10^7} - 7 \cdot \frac{2^{10}}{10^{11}} - \dots\dots \\ &= 5 - 0.008 - 0.0000192 - \dots\dots \end{aligned}$$

The 4th term begins with 7 ciphers; hence this and subsequent terms may be neglected.

Thus $\sqrt[4]{621} = 4.99198 \dots\dots = 4.9920$, to four decimal places.

In these examples the calculation has been made easy by expressing the fractions with powers of 10 in the denominator. Sometimes it will be necessary to proceed as in the next example.

EXAMPLE 3. Find the value of $\frac{1}{\sqrt{50}}$ to five decimal places.

$$\begin{aligned}\frac{1}{\sqrt{50}} &= (50)^{-\frac{1}{2}} = (49 + 1)^{-\frac{1}{2}} = \frac{1}{7} \left(1 + \frac{1}{7^2}\right)^{-\frac{1}{2}} \\ &= \frac{1}{7} \left\{1 - \frac{1}{2} \cdot \frac{1}{7^2} + \frac{3}{8} \cdot \frac{1}{7^4} - \frac{5}{16} \cdot \frac{1}{7^6} + \dots\right\} \\ &= \frac{1}{7} - \frac{1}{2} \cdot \frac{1}{7^3} + \frac{3}{8} \cdot \frac{1}{7^5} - \frac{5}{16} \cdot \frac{1}{7^7} + \dots \\ &= 0.1428571 - 0.0014577 + 0.0000223 - \dots \\ &= 0.14142, \text{ to five decimal places.}\end{aligned}$$

$$\begin{array}{r|l} 7 & 1.00 \dots \\ 7 & \underline{0.1428571} = \frac{1}{7} \\ 7 & \underline{0.0204081} \\ 7 & \underline{0.0029154} = \frac{1}{7^3} \\ 7 & \underline{0.0004165} \\ & 0.0000595 = \frac{1}{7^5} \end{array}$$

The 4th term is neglected since it will not affect the first five decimal figures.

EXAMPLE 4. When x is so small that its square and higher powers may be neglected, find the value of

$$\frac{\sqrt[3]{8+3x} - \sqrt[5]{1-x}}{(1+5x)^{\frac{3}{5}}}.$$

$$\begin{aligned}\text{The expression} &= \frac{(8+3x)^{\frac{1}{3}} - (1-x)^{\frac{1}{5}}}{(1+5x)^{\frac{3}{5}}} = \frac{2\left(1 + \frac{3x}{8}\right)^{\frac{1}{3}} - (1-x)^{\frac{1}{5}}}{(1+5x)^{\frac{3}{5}}} \\ &= \frac{2\left(1 + \frac{1}{8}x + \dots\right) - \left(1 - \frac{1}{5}x + \dots\right)}{(1+3x + \dots)} = \frac{1 + \frac{9}{20}x}{1+3x}, \text{ neglecting } x^2, \\ &= \left(1 + \frac{9}{20}x\right)(1+3x)^{-1} = \left(1 + \frac{9}{20}x\right)(1-3x) \\ &= 1 - \frac{51}{20}x.\end{aligned}$$

[Examples XLI. e. 1-22, page 498, may be taken here.]

542. Suppose $(1+x)^n = 1 + ax + bx^2 + cx^3 + \dots$, where x is small; then $1+ax$ is called a *first approximation*, $1+ax+bx^2$ a *second approximation*, and so on. In a large number of numerical calculations the required accuracy is secured by using the following first approximations which should now be sufficiently obvious.

When r, s, t are small quantities such that their squares, and the products of any two of them may be neglected,

$$(i) (1+r)(1+s) = 1 + (r+s), \quad (1+r)(1-s) = 1 + (r-s).$$

$$\text{Hence} \quad (1+r)^2 = 1 + 2r. \quad \text{Also } (1-r)^2 = 1 - 2r.$$

$$(ii) (1+r)(1+s)(1+t) = 1 + (r+s+t).$$

$$\text{Hence} \quad (1+r)^3 = 1 + 3r. \quad \text{Also } (1-r)^3 = 1 - 3r.$$

$$(iii) \sqrt{1+r} = 1 + \frac{1}{2}r, \quad \sqrt{1-r} = 1 - \frac{1}{2}r.$$

$$(iv) \frac{1}{1+r} = 1 - r, \quad \frac{1}{1-r} = 1 + r.$$

EXAMPLE 1. Find approximately the values of

$$(i) 1.0065 \times .993; \quad (ii) \frac{793}{10004}; \quad (iii) \frac{(1.00027)^3}{(.9991)^2}.$$

$$(i) 1.0065 \times .993 = (1 + .0065)(1 - .007) = 1 + .0065 - .007, \text{ approximately,} \\ = 1 - .0005 = .9995.$$

The error here is the neglected product $.0065 \times .007$, or $.0000455$; thus the result correct to four significant figures is $.9995$.

$$(ii) \text{ Here } 10004 = 10^4 + 4 = 10^4(1 + .0004).$$

$$\therefore \frac{793}{10004} = \frac{793}{10^4(1 + .0004)} = .0793(1 + .0004)^{-1} = .0793(1 - .0004), \text{ approx.,} \\ = .0793 - .0004 \text{ of } .0793 = .0793 - .00003172 = .07927.$$

From this example it may be seen that to divide a number by $10000 + x$, where x is very small, we have only to divide the number by 10000 and subtract x ten-thousandths of the result. Similarly for any divisor of the form $10^n + x$.

$$(iii) (1.00027)^3 = (1 + .00027)^3 = 1 + .00081, \text{ approximately,}$$

$$(.9991)^2 = (1 - .0009)^2 = 1 - .0018,$$

$$\therefore \text{ the required quotient} = \frac{1 + .00081}{1 - .0018} = (1 + .00081)(1 - .0018)^{-1} \\ = (1 + .00081)(1 + .0018) = 1 + .00081 + .0018 \\ = 1 + .00261 = 1.00261.$$

EXAMPLE 2. L and R are the lengths in inches of the arms of an untrue balance. In an experiment by "double-weighing" it was found that $\left(\frac{L}{R}\right)^2 = \frac{49.998}{49.982}$. Find by how much the length of L exceeds that of R .

$$\text{We have } \frac{L}{R} = \sqrt{\frac{49.998}{49.982}} = \left(1 + \frac{.016}{49.982}\right)^{\frac{1}{2}} = \left(1 + \frac{.016}{50}\right)^{\frac{1}{2}}, \text{ approximately,} \\ = 1 + \frac{1}{2}(.00032) = 1 + .00016.$$

Thus L is longer than R by about $.00016$ of either.

EXAMPLE 3. The time t of the beat of a pendulum x centimetres long is $\pi\sqrt{x/981}$; if the pendulum makes n beats in a day, find how many beats it will make if its length is changed to $x + h$ centimetres, where h is very small compared with x .

Let t' be the time of a beat of the lengthened pendulum, and n' the number of beats in a day;

$$\text{then} \quad t' = \pi\sqrt{(x+h)/981}, \text{ and } nt = n't'.$$

$$\therefore \frac{n'}{n} = \frac{t}{t'} = \left(\frac{x}{x+h}\right)^{\frac{1}{2}} = \left(1 + \frac{h}{x}\right)^{-\frac{1}{2}} = 1 - \frac{1}{2} \cdot \frac{h}{x}, \text{ approximately.}$$

Thus $n' = n - \frac{n}{2} \cdot \frac{h}{x}$, so that the longer pendulum loses $\frac{n}{2} \cdot \frac{h}{x}$ beats per day.

EXAMPLE 4. The specific heat s of a metal is found from the equation $ms\theta = Mt$, where m , θ , M , and t are quantities which are determined experimentally. If there may be an error of one per cent. either way in the measurements of m , θ , and t , while the error in M is negligible, find the largest possible percentage error in the value obtained for s .

Since $s = \frac{Mt}{m\theta}$, the percentage error in s will obviously be greatest when the error in t is in either way and the errors in m and θ are in the opposite way.

Let the true values of t , m , θ , and s be represented by t' , m' , θ' , and s' ; then we may put

$$t' = t(1 - .01), \quad m' = m(1 + .01), \quad \theta' = \theta(1 + .01).$$

$$\begin{aligned} \therefore s' &= \frac{Mt'}{m'\theta'} = \frac{Mt(1 - .01)}{m\theta(1 + .01)^2} = s(1 - .01)(1 + .01)^{-2} \\ &= s(1 - .01)(1 - 2 \times .01), \text{ approximately,} \\ &= s(1 - 3 \times .01) = s(1 - .03), \text{ neglecting the square of } .01. \end{aligned}$$

Thus the greatest possible percentage error in s is 3.

EXAMPLES XLI. e

Find to five places of decimals the value of

1. $(1.02)^4$. 2. $(0.97)^5$. 3. $\sqrt{50}$. 4. $\sqrt[3]{1003}$.

Find to four places of decimals the value of

5. $\sqrt[3]{217}$. 6. $\sqrt[5]{30}$. 7. $\sqrt[4]{620}$. 8. $\sqrt[3]{998}$.
9. $\frac{1}{\sqrt{47}}$. 10. $(126)^{-\frac{1}{3}}$. 11. $(626)^{\frac{3}{4}}$. 12. $\sqrt[3]{\frac{251}{250}}$.

13. Find the square root of 2 to six places of decimals.

$$\left[\text{Note that } \sqrt{2} = \sqrt{\frac{100}{49} \cdot \frac{98}{100}} = \frac{10}{7} \left(1 - \frac{1}{50}\right)^{\frac{1}{2}}. \right]$$

When x is so small that its square and higher powers may be neglected, find the value of

14. $(1 + 3x)^{\frac{1}{2}} \cdot (1 - 2x)^{-\frac{1}{3}}$.

15. $(8 + 4x)^{\frac{1}{3}} \cdot (16 - x)^{\frac{1}{4}}$.

16. $\frac{\sqrt{1+x} + \sqrt[3]{(1-x)^2}}{1+x+\sqrt{1+x}}$.

17. $\frac{\sqrt[3]{1+x} + \sqrt[5]{(1-x)^2}}{\sqrt{1+x} + \sqrt[3]{1-x}}$.

18. $\frac{(1+4x)^{\frac{3}{2}} + (1-9x)^{\frac{2}{3}}}{\sqrt{1+7x} + \sqrt[3]{1-3x}}$.

19. $\frac{(1+\frac{3}{2}x)^{-7} + (1-2x)^{-1}}{(1+\frac{7}{2}x)^{-3} + (1+x)^{-2}}$.

20. $\frac{(4+x)^{\frac{3}{2}} \times (8-6x)^{\frac{1}{3}}}{\sqrt[5]{(1+x)^2}}$.

21. $\frac{\sqrt[3]{(8+4x)^2} \times \sqrt[6]{64-36x}}{\sqrt{16-5x} + 4\sqrt{1+2x}}$.

22. Find the first 3 terms in the expansion of $\frac{4}{(2+x)^2 \times \sqrt{1+4x}}$.

Find by the formulæ of Art. 542 the approximate values of

23. 1.0078×0.9954 . 24. $(0.9987)^2$. 25. $(1.005) \times (1.0002)^2$.
26. 1.00039×0.99978 . 27. $1.00064 \div 1.0078$.
28. $(1.00018)^2 \times (1.00004)^3$. 29. $1.0058 \div 0.9982$.
30. $\frac{678}{1003}$. 31. $\frac{1}{0.9998}$. 32. $\frac{1.007}{(0.995)^2}$.
33. $\frac{1}{(0.98)^3}$. 34. $\frac{1}{\sqrt{0.997}}$. 35. $\frac{\sqrt{0.9987}}{(1.0003)^2}$.
36. In three consecutive years the population of a town was increased by 0.8%, 1.2%, 2.5%. Find approximately the total percentage increase in the three years.
37. A bar of copper increases by 0.0000168 of its length for a rise of temperature of 1 degree Centigrade. Find the expansion in area of a rectangular copper plate 45 cm. long and 33 cm. wide when the temperature is raised 10 degrees, the metal being assumed to expand equally in all directions. [The decimal 0.0000168 is called the **coefficient of linear expansion** for copper.]
38. The coefficient of linear expansion for zinc is 0.0000298. If the area of a zinc roof is 1000 square feet at 0° C., find its area to the nearest square foot at 50° C.
39. If $\left(\frac{x}{y}\right)^2 = \frac{9.98}{9.92}$, shew that x exceeds y by about 0.3%.
40. If $t^2 = \frac{\pi^2 l}{g}$, and $T^2 = \frac{\pi^2 L}{g}$, and L is very nearly equal to l , shew that $\frac{T-t}{t} = \frac{L-l}{2l}$, very nearly. Find the error in this equation to the first significant figure when $l=36$ and $L-l=0.1$.
41. The time of one beat of a pendulum, l feet in length, is given by the formula $\pi\sqrt{\frac{l}{32.2}}$. How many beats will a "seconds pendulum" gain in 10 hours if its length is shortened by 2%?
42. The value of P has to be found from the formula $P = \frac{k \times t^2}{d \times \sqrt{l}}$ where k is a constant, while t , d , and l are found by experiment. Find the percentage error in the value of P , supposing there is an error of 0.4% in the value of t , and an error of 1.0% in the value of l , the former error being in excess and the latter in defect.
43. If in the formula of Ex. 42, there is an error of 0.3% too little in the observed value of t , an error of 1.0% too much in the value of d , and of 2.0% too much in the value of l , find the percentage error in the value of P .

44. The modulus of torsion of a wire is found from the formula $n = \frac{2\pi Il}{t^2 r^4}$.

If there are errors of 0.2% too little in the value of l , 1.0% too much in the value of t , and 2.5% too little in the value of r , find the resulting error per cent. in the value of n .

(Miscellaneous)

45. Express the 10th term of $(1+x)^{\frac{1}{2}}$ with a numerator containing the product of odd numbers, and a denominator a power of 2.

46. Find the coefficient of x^3 in the expansion of $(1+x-6x^2)^n$.

47. Find the first 3 terms in the expansion of $\frac{1}{(2-x)(2-3x)^{\frac{3}{2}}}$.

48. In the expansion of $\frac{3+x^2}{(1+x)^3}$ find the coefficient (i) of x^5 , (ii) of x^{2n+1} ; also deduce the first result from the second.

49. Find the coefficient of x^n in the expansion of $(1-4x)^{-\frac{1}{2}}$, and shew that it is equal to the term independent of x in $\left(x + \frac{1}{x}\right)^{2n}$.

50. Shew that $\sqrt{x^2+4} - \sqrt{x^2+1}$ is equal to

$$1 - \frac{1}{4}x^2 + \frac{7}{64}x^4 - \dots \quad \text{or} \quad \frac{3}{2x} \left\{ 1 - \frac{5}{4x^2} + \frac{21}{8x^4} - \dots \right\},$$

according as x is less or greater than 1.

51. If b is so small compared with a that $b^2, b^3, \dots$ may be neglected, find the sum of n terms of the H.P.

$$\frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \dots$$

52. If $\sqrt{N} = a + x$, where x is very small compared with a , shew that

$$\sqrt{N} = a \cdot \frac{3N + a^2}{N + 3a^2}, \text{ very nearly.}$$

Test the formula when $N = 10, 26, 123$.

[We have $N = a^2 + 2ax + x^2$. As a first approximation $N = a^2 + 2ax$, whence $x = \frac{N - a^2}{2a}$, and $2a + x = \frac{N + 3a^2}{2a}$. Substitute this value in $N = a^2 + (2a + x)x$; thence find x and substitute in $\sqrt{N} = a + x$.]

53. If $\sqrt[n]{N} = a + x$, where x is very small compared with a , shew by taking three terms of the expansion of $(a+x)^n$ that

$$\sqrt[n]{N} = a \cdot \frac{(n+1)N + (n-1)a^n}{(n-1)N + (n+1)a^n}, \text{ approximately.}$$

CHAPTER XLII

PARTIAL FRACTIONS

543. THE algebraic sum of two or more algebraic proper fractions can always be expressed in the form of a single proper fraction.

Thus
$$\frac{3}{x-1} - \frac{2}{x-3} = \frac{x-7}{(x-1)(x-3)},$$

$$\frac{1}{1-x} + \frac{2}{1+x} + \frac{2}{1-3x} = \frac{5-10x+x^2}{(1-x)(1+x)(1-3x)}.$$

We shall now explain the converse process, namely that of resolving a fraction into a group of simpler fractions, each having for its denominator one of the factors of the denominator of the given fraction. Such fractions are known as **partial fractions**. The resolution is effected by the method of Undetermined Coefficients, and some easy cases have already occurred. See Art. 475, Ex. 2.

In each of the above examples we notice that the single fraction on the right has a numerator of lower dimensions than the denominator. We may assume this to be the case in the fractions given for resolution into partial fractions. For if the numerator of a fraction is not of lower dimensions than the denominator, by division the fraction can be expressed in a form partly integral and partly fractional, the latter part having a numerator of lower dimensions than the denominator.

EXAMPLE 1. Resolve $\frac{x+16}{(2x-3)(x+2)}$ into partial fractions.

The denominators $2x-3$ and $x+2$ being of the first degree, we may assume

$$\frac{x+16}{(2x-3)(x+2)} = \frac{A}{2x-3} + \frac{B}{x+2}, \text{ where } A \text{ and } B \text{ are constants.}$$

Multiplying both sides by $(2x-3)(x+2)$, we have

$$\begin{aligned} x+16 &\equiv A(x+2) + B(2x-3), \dots\dots\dots(1) \\ &\equiv (A+2B)x + (2A-3B). \end{aligned}$$

Since this is identically true, by equating coefficients we have

$$A+2B=1, \quad 2A-3B=16;$$

whence

$$\begin{aligned} A &= 5, \quad B = -2; \\ \therefore \frac{x+16}{(2x-3)(x+2)} &= \frac{5}{2x-3} - \frac{2}{x+2}. \end{aligned}$$

NOTE. Since (1) is true for all values of x , we may also find A and B by giving different numerical values to x in this identity.

EXAMPLE 2. Resolve $\frac{2x^3 + 7x^2 - 2x - 2}{2x^2 + x - 6}$ into partial fractions.

Here the numerator is not of lower dimensions than the denominator; hence we proceed as follows:

$$\begin{aligned} \text{by division, } \frac{2x^3 + 7x^2 - 2x - 2}{2x^2 + x - 6} &= x + 3 + \frac{x + 16}{(2x - 3)(x + 2)} \\ &= x + 3 + \frac{5}{2x - 3} - \frac{2}{x + 2}, \text{ by Ex. 1.} \end{aligned}$$

EXAMPLE 3. Resolve $\frac{3x^2 - 10x - 2}{(x - 1)(x - 2)(2x + 1)}$ into partial fractions.

$$\text{Assume } \frac{3x^2 - 10x - 2}{(x - 1)(x - 2)(2x + 1)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{2x + 1};$$

$$\text{then } 3x^2 - 10x - 2 \equiv A(x - 2)(2x + 1) + B(x - 1)(2x + 1) + C(x - 1)(x - 2).$$

$$\text{Put } x = 1; \text{ then } 3 - 10 - 2 = -3A, \text{ or } A = 3.$$

$$\text{Put } x = 2; \text{ then } 12 - 20 - 2 = 5B, \text{ or } B = -2.$$

$$\text{To find } C, \text{ equate the coefficients of } x^2, \text{ then } 3 = 2A + 2B + C;$$

$$\text{whence } 3 = 6 - 4 + C, \text{ or } C = 1.$$

$$\text{Thus } \frac{3x^2 - 10x - 2}{(x - 1)(x - 2)(2x + 1)} = \frac{3}{x - 1} - \frac{2}{x - 2} + \frac{1}{2x + 1}.$$

EXAMPLE 4. Express $\frac{2x^2 - 5x + 10}{(x + 2)(x^2 + x + 5)}$ in partial fractions.

Since $x^2 + x + 5$ has no rational factors, we shall have two fractions with denominators $x + 2$ and $x^2 + x + 5$. The numerator of the first will be constant as before, but the numerator of the second may be of the first degree in x . Hence we assume

$$\frac{2x^2 - 5x + 10}{(x + 2)(x^2 + x + 5)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 + x + 5}, \text{ where } A, B, C \text{ are constants.}$$

$$\therefore 2x^2 - 5x + 10 \equiv A(x^2 + x + 5) + (x + 2)(Bx + C)$$

$$\equiv (A + B)x^2 + (A + 2B + C)x + (5A + 2C).$$

By equating coefficients, we have

$$A + B = 2, \quad A + 2B + C = -5, \quad 5A + 2C = 10.$$

These three equations give $A = 4, B = -2, C = -5$.

$$\therefore \frac{2x^2 - 5x + 10}{(x + 2)(x^2 + x + 5)} = \frac{4}{x + 2} - \frac{2x + 5}{x^2 + x + 5}.$$

NOTE. If we had assumed $\frac{2x^2 - 5x + 10}{(x + 2)(x^2 + x + 5)} = \frac{A}{x + 2} + \frac{C}{x^2 + x + 5}$, leaving out the term Bx , the next step would have given

$$2x^2 - 5x + 10 \equiv Ax^2 + (A + C)x + (5A + 2C),$$

and on equating coefficients we should have had

$$A = 2, \quad A + C = -5, \quad 5A + 2C = 10,$$

and these three equations will be found to be inconsistent.

EXAMPLE 5. Resolve $\frac{7x-10}{(3x-4)(x-1)^2}$ into partial fractions.

As before, we may assume the fraction $\equiv \frac{A}{3x-4} + \frac{Bx+B'}{(x-1)^2}$.

$$\text{Now } \frac{Bx+B'}{(x-1)^2} = \frac{B(x-1) + (B+B')}{(x-1)^2} = \frac{B}{x-1} + \frac{B+B'}{(x-1)^2};$$

$\therefore$ we may at once assume

$$\frac{7x-10}{(3x-4)(x-1)^2} \equiv \frac{A}{3x-4} + \frac{B}{x-1} + \frac{C}{(x-1)^2}, \text{ where } A, B, C \text{ are constants.}$$

$$\therefore 7x-10 \equiv A(x-1)^2 + B(3x-4)(x-1) + C(3x-4).$$

Put $x-1=0$; then $7-10=-C$, or $C=3$.

By equating coefficients, $A+3B=0$, and $-10=A+4B-4C$;

whence $A=-6, B=2$.

$$\therefore \frac{7x-10}{(3x-4)(x-1)^2} = -\frac{6}{3x-4} + \frac{2}{x-1} + \frac{3}{(x-1)^2}.$$

Similarly, corresponding to a factor of the form $(x-a)^r$ in the denominator, we may assume fractions with constant numerators, and denominators $(x-a), (x-a)^2, \dots (x-a)^r$.

[Examples XLII. 1-12, page 504, may be taken here.]

544. The following example shews a useful application of partial fractions.

EXAMPLE. Find the general term in the expansion of $\frac{7x-10}{(3x-4)(x-1)^2}$ in ascending powers of x .

$$\text{We have } \frac{7x-10}{(3x-4)(x-1)^2} = -\frac{6}{3x-4} + \frac{2}{x-1} + \frac{3}{(x-1)^2} \quad [\text{Ex. 5, Art. 543.}]$$

$$\begin{aligned} &= \frac{6}{4 \left(1 - \frac{3x}{4}\right)} - \frac{2}{1-x} + \frac{3}{(1-x)^2} \\ &= \frac{3}{2} \left(1 - \frac{3x}{4}\right)^{-1} - 2(1-x)^{-1} + 3(1-x)^{-2} \\ &= \frac{3}{2} \left\{ 1 + \frac{3x}{4} + \left(\frac{3x}{4}\right)^2 + \dots + \left(\frac{3x}{4}\right)^r + \dots \right\} \\ &\quad - 2 \{ 1 + x + x^2 + \dots + x^r + \dots \} \\ &\quad + 3 \{ 1 + 2x + 3x^2 + \dots + (r+1)x^r + \dots \}. \end{aligned}$$

Thus the general term

$$\begin{aligned} &= \left\{ \frac{3}{2} \cdot \frac{3^r}{4^r} - 2 + 3(r+1) \right\} x^r \\ &= \left(\frac{3^{r+1}}{2^{2r+1}} + 3r + 1 \right) x^r. \end{aligned}$$

545. In some special cases a fraction may be resolved into partial fractions very readily without the use of Undetermined Coefficients.

EXAMPLE. Separate $\frac{x^3 + 8}{(x - 2)^4}$ into partial fractions.

Put $x - 2 = y$, so that $x = y + 2$;

$$\begin{aligned} \text{then } \frac{x^3 + 8}{(x - 2)^4} &= \frac{(y + 2)^3 + 8}{y^4} = \frac{y^3 + 6y^2 + 12y + 16}{y^4} \\ &= \frac{1}{y} + \frac{6}{y^2} + \frac{12}{y^3} + \frac{16}{y^4} \\ &= \frac{1}{x - 2} + \frac{6}{(x - 2)^2} + \frac{12}{(x - 2)^3} + \frac{16}{(x - 2)^4}. \end{aligned}$$

EXAMPLES XLII

Resolve into partial fractions :

- | | | |
|---|---|--|
| 1. $\frac{5x + 1}{(x + 5)(x - 3)}$ | 2. $\frac{4x - 19}{(x - 1)(x - 2)}$ | 3. $\frac{14x}{x^2 + x - 12}$ |
| 4. $\frac{8 - x}{1 + x - 6x^2}$ | 5. $\frac{x^2 - 6x - 7}{(x - 1)(x - 2)(x + 3)}$ | 6. $\frac{x^2 + 8x + 1}{(2 - x)(1 + x + x^2)}$ |
| 7. $\frac{3x^2 + x + 1}{x(x + 1)^2}$ | 8. $\frac{x^2 - 2x + 10}{(x + 2)(x - 1)^2}$ | 9. $\frac{x^3 - 2x - 13}{x^2 - 2x - 3}$ |
| 10. $\frac{x^2 + 4x + 7}{(x + 2)(x + 3)^2}$ | 11. $\frac{3x^2 - x - 2}{(1 + 2x)(x + 2)^2}$ | 12. $\frac{3x^2 + 92x}{(x^2 + 1)(x + 6)}$ |

Find the general term when the following expressions are expanded in ascending powers of x .

- | | | |
|--|--|---|
| 13. $\frac{3 - 4x}{(1 - x)(1 - 2x)}$ | 14. $\frac{2}{x^2 - 8x + 15}$ | 15. $\frac{7 + 6x}{2 + 7x + 3x^2}$ |
| 16. $\frac{2 + 13x}{2x^2 - x - 1}$ | 17. $\frac{2x - 4}{(1 - x^2)(1 - 2x)}$ | 18. $\frac{x^2}{(x^2 - 1)(x - 2)}$ |
| 19. $\frac{3 - 2x - x^2}{(1 - x)(1 + 4x)^2}$ | 20. $\frac{4 + 7x}{(2 + 3x)(1 + x)^2}$ | 21. $\frac{x^2}{(x - 1)^2(x - 2)(x - 3)}$ |
| 22. Express $\frac{x^2 - 6x + 5}{(x - 4)^3}$ and $\frac{x^3 + 3x^2 - 2x - 4}{(x + 2)^4}$ in partial fractions. | | |

23. Express $\frac{1}{(3n - 2)(3n + 1)}$ in partial fractions ; thence find the sum of

$$\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \dots \text{ to } n \text{ terms.}$$

MISCELLANEOUS EXAMPLES IX

EXAMPLES FOR REVISION

A

1. A horse bought for £ x is sold for £39 at a profit of x per cent. Find x .

2. Given that $y = \frac{3-2z}{4z+1}$ and $z = \frac{5+4x}{3x-2}$, find the value of y in terms of x , and of x in terms of y .

3. If $x-2$ is a factor of $120x^3 - 167x^2 - ax + 56$, find the numerical value of a , and hence find the two remaining factors of the expression.

4. The first and last terms of an A.P. are -3 and 25 , and the sum of the series is 1837 . Find the number of terms and the common difference.

5. Find the coefficient of x in the expansion of $\left(x^{13} - \frac{3}{x^5}\right)^7$.

6. Solve the equations :

$$2(1-x) + (2-y) = 3(1-x)(2-y),$$

$$4(1-x) + (2-y) = 9(1-x)(2-y).$$

7. Draw the graph of $y = \frac{8}{9} + \frac{4}{3}x - x^2$, and from the graph, or otherwise, determine the value of x for which y is a maximum.

Shew that the line whose equation is $y = 2x + 1$ touches the above curve, and write down the coordinates of the point of contact.

B

8. Find the factors of

$$(i) (m^2 - n^2)t - mn(t^2 - 1); \quad (ii) x^4 - 7x^2y^2 + y^4.$$

9. A man has £ a in one bank and £ b in another. If he has £ c more to deposit, how should he divide it between the two banks so that (i) the two accounts may be equal, (ii) the first account may be double the second ?

10. From the statement $ax^2 + bx = cx - \frac{1}{d}$, find

(i) d when $a=1$, $b=2$, $c=-1$, $x=-2$;

(ii) a and c when $x=0.4$, $b=3$, $d=-1$, and $a+c=2.25$.

11. A gallon of water weighs 10 lbs., and a gallon of milk 10.32 lbs. ; if a milkman sells a mixture of 10 gallons weighing 102.5 lbs., how much milk is there in the mixture ?

12. If l and m are real quantities, shew that the expression

$$x^2 - (l+m)x + l^2 - lm + m^2$$

will be positive for all real values of x .

13. If $A - 24y + 10y^2 + 8y^3 + y^4$ is an exact square, what is the value of A ?

14. Find the number of arrangements of the expanded form of the expression $a^m b^2$, such that the two b 's come together in each arrangement. Prove also that the number of arrangements in which the two b 's never come together is $\frac{m(m+1)}{2}$.

C

15. If $x + \frac{1}{x} = y$, prove that $x^5 + \frac{1}{x^5} = y^5 - 5y^3 + 5y$.

16. If p dozen oranges worth b pence per score are selected from a case containing q gross of average value 23 for a shilling, what is the average value per dozen of the oranges which are left in the case?

17. The time of swing of a pendulum is equal to $k\sqrt{l}$ seconds, where l is the length of the pendulum in inches and k a constant. The length of a pendulum beating seconds is 39.14 inches; hence find the time of swing of a pendulum 60 inches long.

18. With the same axes of x and y draw the graphs of

$$(i) y = 2x - \frac{1}{3}x^2; \quad (ii) y = \frac{1}{2}x - 2.$$

Write down the quadratic equation whose roots are the values of x at the points of intersection of the graphs.

19. Find the coefficient of x^r in the expansion of $\frac{5+4x}{(1-x)^2}$.

20. Shew that $x^2 + qx + 1$ and $x^3 + px^2 + qx + 1$ have a common factor of the form $x + a$ when

$$(p-1)^2 - q(p-1) + 1 = 0.$$

21. If $\frac{x^2 + 15x - 22}{(x-1)(x-2)(x+2)} \equiv \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+2}$, find A , B , and C .

D

22. An alloy, whose value is $\text{£}a$ per lb., is composed of two metals; one of the metals, worth $\text{£}b$ per lb. is extracted; this metal was contained in the alloy in the ratio $x:1$. What is the value of the other metal per lb.?

23. Solve the equations:

$$(i) \frac{2x+3}{4x+3} + \frac{3x+2}{4x+2} = \frac{5(x+2)}{4x+5}; \quad (ii) x^2 + x \left(\frac{p^2}{q} - \frac{q^2}{p} \right) - pq = 0.$$

24. Find the sum of n terms of the geometric progression of which the third term is -24 and the sixth term is 3 .

How many terms must be taken so that this sum may differ from the sum to infinity by less than 0.001 ?

25. If x is so small that its cube and higher powers may be neglected, prove that

$$\frac{(1-4x)^{\frac{1}{2}} + (1-3x)^{\frac{1}{3}}}{(1-2x)^{\frac{1}{4}}} = 2 - 2x - \frac{13x^2}{4}.$$

26. If a, b, c, d, x are positive, and $\frac{a}{b} > \frac{c}{d}$, prove that

$$\frac{a}{b} > \frac{a+xc}{b+xd} > \frac{c}{d}.$$

27. Calculate the values of $(\sqrt{2})^x$ for $x=0, 1, 2, 3, 4, 5, 6$, and plot the results in a graph.

28. If $a^2 = q + r$, $b^2 = r + p$, $c^2 = p + q$, and $2s = a + b + c$, prove that

$$4s(s-a)(s-b)(s-c) = qr + rp + pq.$$

E

29. Simplify (i) $\frac{3^{2n} - 3 \cdot 9^{n-1}}{6^{2n-1}}$; (ii) $\frac{1}{\sqrt{11-2\sqrt{30}}} - \frac{3}{\sqrt{7-\sqrt{40}}}$.

30. If $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ are the roots of the equation $px^2 - 2x + p^2 = 0$, prove that $\alpha = \pm\beta$.

31. If $ax + by = 0$, $x + y = xy$, and $x^2 + y^2 = 1$, prove that

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{(a-b)^2}.$$

32. If n straight lines of unlimited length, p of which are parallel, are drawn in a plane, prove that the number of triangles so formed is

$$\frac{1}{6}(n-p)(n-\overline{p+1})\{n+2(p-1)\},$$

supposing that no three of the lines are concurrent.

33. What is the error made in taking the sum of the infinite series 1, 0.2, 0.04, 0.008, as being 1.248?

Find, to three decimal places, the sum of the square roots of the terms of this series (i) taken as all positive, (ii) taken as alternately positive and negative.

34. Running a certain course, starting level, A finishes 176 yards in front of B . With a minute's start, B would have finished 160 yards in front of A ; and with 160 yards start he would have finished three seconds later than A . What is the length of the course, and what are the times of A and B for the full course?

35. Draw a graph of the expression $\left(\frac{x}{3}\right)^2(6-x)$, using the values $x=0, 0.5, 1.0, 1.5, \dots, 5.5, 6.0$.

F

36. Solve the equation $\sqrt{x-a} + \sqrt{x-b} = \sqrt{a+b}$, and verify the solution.
37. Find p so that the sum of the squares of the roots of the equation $(p-6)x^2 + (12-p)x + 3p = 0$ may be equal to 4.
38. Find $\sqrt{97+56\sqrt{3}}$ in the form $x + \sqrt{y}$, and find the square root of the last surd.
39. If $2ax = a^2 + 1$, $2by = b^2 + 1$, and $xy - \sqrt{(x^2-1)(y^2-1)} = c$,
 prove that
$$\sqrt{\frac{c+1}{c-1}} = \frac{a+b}{a-b}.$$
40. The two middle terms of an A.P. of $2n$ terms are a and b . Find the difference between the sum of the last n terms and that of the first n terms.
41. The coefficients of the 5th, 6th, and 7th terms in the expansion of $(1+x)^n$ are in arithmetic progression. Find n .
42. Trace, with the same axes, the graphs of

$$y = 0.3x^2 - 1.2, \quad y = x^3.$$
 Hence solve the equation $x^3 = 0.3x^2 - 1.2$.

G.

43. Solve the equations :

$$ax - by = \frac{1}{2}(b-a), \quad ax + by = c(1+z), \quad by - cz = \frac{1}{2}(c-b).$$
44. Find a geometric series such that each term exceeds by unity the sum of all the terms before it.
45. Find a quantity such that when it is subtracted from each of the quantities a, b, c , the remainders are in continued proportion.
46. If $a+b+c=0$, prove that
$$\frac{a^2}{2a^2+bc} + \frac{b^2}{2b^2+ac} + \frac{c^2}{2c^2+ab} = 1.$$
47. Find the number of different permutations that can be made out of the letters a, b, c, d, b, a , taken three at a time.
48. If the increase in a tree's girth in one year is proportional to its girth at the beginning of the year, and its girth is doubled in 11 years, in how many years will its girth be trebled?
49. Find the value of the infinite series

$$1 + \frac{1}{6} + \frac{1.3}{6.12} + \frac{1.3.5}{6.12.18} + \dots$$

CHAPTER XLIII

THE USE OF EXPONENTIAL AND LOGARITHMIC SERIES

546. IN this chapter we shall deal with the use of certain expansions known as Exponential and Logarithmic Series. Rigorous proofs of these expansions do not fall within the scope of an elementary text-book, but the student may conveniently here learn some of the applications of such series, postponing their formal discussion to a later stage of his reading.

547. The infinite series

$$1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{r} + \dots$$

is always denoted by the symbol e .

The numerical value of e is obviously greater than 2.

Also

$$\begin{aligned} e &< 1 + \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \text{ to inf.} \right) \\ &< 1 + \frac{1}{1 - \frac{1}{2}} \\ &< 3. \end{aligned}$$

Thus the value of e lies between 2 and 3. By taking a sufficient number of terms of the expansion, and expressing them in decimal form, the value of e can be obtained to any required degree of accuracy. To six places of decimals it is found to be 2.718282.

548. The series which we have denoted by e is very important as it is the base to which logarithms are first calculated. Logarithms to this base are known as the Napierian system, so named after Napier their discoverer. They are also called *natural* logarithms from the fact that they are the first logarithms which naturally come into consideration in algebraical investigations.

When logarithms are used in theoretical work it is to be remembered that the base e is always understood, just as in arithmetical work *common* logarithms to the base 10 are invariably employed.

The connection between natural and common logarithms will be explained later.

Exponential Series

549. When x has any finite value, it can be proved that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^r}{r} + \dots \text{ to inf.}$$

and that the series on the right is convergent. [See Art. 535.]

This is known as the **Exponential Theorem**.

This theorem can be put in another form, as follows:

Write cx in the place of x ; then

$$e^{cx} = 1 + cx + \frac{c^2x^2}{2} + \frac{c^3x^3}{3} + \dots$$

Now let $e^c = a$, so that $c = \log_e a$, then we obtain

$$a^x = 1 + x \log_e a + \frac{x^2 (\log_e a)^2}{2} + \frac{x^3 (\log_e a)^3}{3} + \dots$$

EXAMPLE 1. Find the coefficient of x^r in the expansion of $\frac{a - bx}{e^x}$.

$$\frac{a - bx}{e^x} = (a - bx)e^{-x}$$

$$= (a - bx) \left\{ 1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \dots + \frac{(-1)^r x^r}{r} + \dots \right\};$$

$$\therefore \text{ the coefficient of } x^r = \frac{(-1)^r}{r} \cdot a - \frac{(-1)^{r-1}}{r-1} \cdot b = \frac{(-1)^r}{r} (a + rb).$$

EXAMPLE 2. Find the sum of the infinite series

$$\frac{1 \cdot 2}{1} + \frac{2 \cdot 3}{2} + \frac{3 \cdot 4}{3} + \frac{4 \cdot 5}{4} + \dots$$

If we denote the successive terms by $u_1, u_2, u_3, \dots$, we have

$$u_n = \frac{n(n+1)}{n} = \frac{n+1}{n-1} \dots \dots \dots (1)$$

$$= \frac{(n-1)+2}{n-1} = \frac{1}{n-2} + \frac{2}{n-1}, \dots \dots \dots (2)$$

Putting $n = 1, 2, 3, \dots$ successively, from (1) we have $u_1 = 2$. And from (2) we have

$$u_2 = 1 + \frac{2}{1}, \quad u_3 = \frac{1}{1} + \frac{2}{2}, \quad u_4 = \frac{1}{2} + \frac{2}{3}, \quad u_5 = \frac{1}{3} + \frac{2}{4}, \text{ and so on.}$$

Hence, by collecting terms suitably,

$$\begin{aligned} \text{the series} &= \left(2 + \frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \dots \right) + \left(1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \right) \\ &= 2e + e = 3e. \end{aligned}$$

[Examples XLIII. 1-11, page 516, may be taken here.]

Logarithmic Series

550. From the Exponential Theorem the following formula can be deduced :

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r-1} \frac{x^r}{r} + \dots$$

This is known as the **Logarithmic Series**.

In this formula the number of terms on the right is infinite, and the series is convergent when x is *greater than* -1 and *not greater than* $+1$. Hence within this range of values the series may be legitimately used for arithmetical calculation.

The form of the series should be carefully noted and compared with that of the exponential series

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^r}{r} + \dots$$

In the latter the first term is 1, all the terms are positive, all the denominators are factorials, and x^r occurs in the $(r+1)^{\text{th}}$ term.

In the logarithmic series the first term is x , the terms are alternately positive and negative, there are no factorials in the denominators, and x^r occurs in the r^{th} term.

The following examples are given to enforce these points. In each case it is assumed that the symbols are of such value as to make the expansions legitimate.

EXAMPLE 1. If $a = b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \dots$, express b in ascending powers of a .

From the given result we have $a = \log_e(1+b)$;

$$\therefore e^a = 1+b, \quad \text{or} \quad b = e^a - 1.$$

Hence

$$b = a + \frac{a^2}{2} + \frac{a^3}{3} + \frac{a^4}{4} + \dots$$

EXAMPLE 2. Shew that

$$\log_e(1+3x+2x^2) = 3x - \frac{5x^2}{2} + 3x^3 - \frac{17x^4}{4} + \dots,$$

and find the general term of the series.

$$\log_e(1+3x+2x^2) = \log_e(1+x)(1+2x) = \log_e(1+x) + \log_e(1+2x)$$

$$\begin{aligned} &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \right) + \left(2x - \frac{4x^2}{2} + \frac{8x^3}{3} - \frac{16x^4}{4} + \dots \right) \\ &= 3x - \frac{5x^2}{2} + 3x^3 - \frac{17x^4}{4} + \dots \end{aligned}$$

$$\text{The general term} = (-1)^{r-1} \frac{x^r}{r} + (-1)^{r-1} \frac{2^r x^r}{r} = \frac{(-1)^{r-1}}{r} (2^r + 1) x^r.$$

Construction of Logarithmic Tables

551. We shall now give some account of the way in which logarithmic series are used in the calculation of Napierian Logarithms, leading up to the construction of Tables of Common Logarithms.

The series for $\log_e(1+x)$ cannot be used for values of $x > 1$; moreover, it converges so slowly that it is of little use for numerical calculations. We can, however, deduce from it other series by the aid of which Tables of Logarithms may be constructed.

We have
$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots;$$

replacing x by $-x$, we have

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots.$$

$$\begin{aligned} \therefore \log_e \frac{1+x}{1-x} &= \log_e(1+x) - \log_e(1-x) \\ &= 2 \left\{ x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \right\} \dots\dots\dots(1) \end{aligned}$$

In this result put $\frac{1+x}{1-x} = \frac{n+1}{n}$, so that $x = \frac{1}{2n+1}$; then

$$\log_e \frac{n+1}{n} = 2 \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots \right\};$$

that is,

$$\log_e(n+1) - \log_e n = 2 \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots \right\} \dots\dots(2)$$

552. In Art. 407 it was proved that to transform logarithms from any base a to a new base b , we have to multiply them by the *modulus* $\frac{1}{\log_a b}$. Hence logarithms of numbers to base 10 can be obtained by multiplying the Napierian logarithms of these numbers by the modulus $\frac{1}{\log_e 10}$.

From the series (2) of the preceding article, we can obtain $\log_e 2$ by putting $n=1$. Again, by putting $n=2$, we obtain $\log_e 3 - \log_e 2$; whence $\log_e 3$ is found, and therefore also $2 \log_e 3$ or $\log_e 9$ is known.

Now by putting $n=9$ in series (2), we can obtain $\log_e 10 - \log_e 9$; whence the value of $\log_e 10$ is found to be $2.30258509 \dots$

Thus the modulus for the system of common logarithms is $\frac{1}{2.30258509 \dots}$, or $0.43429448 \dots$; we shall denote this modulus by μ .

By multiplying the series (2) of Art. 551 throughout by μ we obtain a formula adapted to the calculation of common logarithms. Thus

$$\mu \log_e (n+1) - \mu \log_e n = 2\mu \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots \right\};$$

that is,

$$\log_{10} (n+1) - \log_{10} n = 2 \left\{ \frac{\mu}{2n+1} + \frac{\mu}{3(2n+1)^3} + \frac{\mu}{5(2n+1)^5} + \dots \right\}.$$

From this result we see that if the logarithm of one of two consecutive numbers is known, the logarithm of the other may be found, and thus a table of logarithms can be constructed.

It should be noticed that the above formula is only needed to calculate the logarithms of *prime* numbers, for the logarithm of a *composite* number may be obtained by adding together the logarithms of its component factors.

EXAMPLE. Calculate the value of $\log_{10} 2$ to six decimal places.

Putting $n=1$ in the last series, we have

$$\log_{10} 2 = 2 \left\{ \frac{\mu}{3} + \frac{\mu}{3 \cdot 3^3} + \frac{\mu}{5 \cdot 3^5} + \frac{\mu}{7 \cdot 3^7} + \dots \right\}.$$

The calculation may be arranged as follows :

$\mu = .43429448,$	$\therefore \mu/3$	$= .144764$	83
$\mu/3^3 = \mu/3 \div 9 = .01608498,$	$\therefore \mu/(3 \cdot 3^3)$	$= .005361$	66
$\mu/3^5 = .00178722,$	$\therefore \mu/(5 \cdot 3^5)$	$= .000357$	44
$\mu/3^7 = .00019858,$	$\therefore \mu/(7 \cdot 3^7)$	$= .000028$	37
$\mu/3^9 = .00002206,$	$\therefore \mu/(9 \cdot 3^9)$	$= .000002$	45
$\mu/3^{11} = .00000245,$	$\therefore \mu/(11 \cdot 3^{11})$	$= .000000$	22
$\mu/3^{13} = .00000027,$	$\therefore \mu/(13 \cdot 3^{13})$	$= .000000$	02
		<u>.150515</u>	

$$\therefore \log_{10} 2 = .150515 \times 2 = .301030.$$

553. Theoretically the series for $\log_{10}(n+1) - \log_{10} n$ given in the last article is sufficient for the calculation of common logarithms. It has the advantage of converging rapidly, so that (except for small values of n) only a few terms of the series need be taken to obtain the necessary approximation; but in practice the arithmetical work is often inconvenient.

For example, when $n=16$, we get $\log_{10} 17 - \log_{10} 16$;

$$\text{that is, } \log_{10} 17 = 4 \log_{10} 2 + 2 \left\{ \frac{\mu}{33} + \frac{\mu}{3(33)^3} + \frac{\mu}{5(33)^5} + \dots \right\},$$

and the calculation of the terms of the series would be very tedious. We shall now give other series which will effect a saving of labour.

554. In the formula

$$\log_{10}(1+x) = \mu \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \right),$$

by writing $\frac{1}{n}$ for x , we obtain $\log_{10} \frac{n+1}{n}$; hence

$$\log_{10}(n+1) - \log_{10} n = \frac{\mu}{n} - \frac{\mu}{2n^2} + \frac{\mu}{3n^3} - \dots \quad (1)$$

Again, by writing $-\frac{1}{n}$ for x , we obtain $\log_{10} \frac{n-1}{n}$; hence by changing signs on both sides of the formula, we have

$$\log_{10} n - \log_{10}(n-1) = \frac{\mu}{n} + \frac{\mu}{2n^2} + \frac{\mu}{3n^3} + \dots \quad (2)$$

The following example shews the use of these series in obtaining the logarithms of some of the smaller prime numbers. It will be seen that the calculation is usually less laborious than in the example of Art. 552. The numerical details are left as an exercise for the student.

EXAMPLE. *To explain how the common logarithms of 2, 3, 5, 7, 11 may be found.*

(i) Putting $n=10$ in series (2), we get $\log 10 - \log 9$; thus

$$1 - 2 \log 3 = \frac{\mu}{10} + \frac{\mu}{2 \cdot 10^2} + \frac{\mu}{3 \cdot 10^3} + \dots;$$

whence $\log 3$ is readily found to be $\cdot 4771213$, to seven decimal places.

(ii) By putting $n=3$ in series (2), we get $\log 3 - \log 2$; thus

$$\log 3 - \log 2 = \frac{\mu}{3} + \frac{\mu}{2 \cdot 3^2} + \frac{\mu}{3 \cdot 3^3} + \dots;$$

whence $\log 2$ is found to be $\cdot 3010300$.

(iii) $\log 5 = \log \frac{10}{2} = 1 - \log 2 = \cdot 6989700$.

(iv) By putting $n=8$ in series (2), we get $\log 8 - \log 7$; thus

$$3 \log 2 - \log 7 = \frac{\mu}{8} + \frac{\mu}{2 \cdot 8^2} + \frac{\mu}{3 \cdot 8^3} + \dots;$$

whence $\log 7$ is found to be $\cdot 8450980$.

(v) By putting $n=10$ in series (1), we get $\log 11 - \log 10$; thus

$$\log 11 - 1 = \frac{\mu}{10} - \frac{\mu}{2 \cdot 10^2} + \frac{\mu}{3 \cdot 10^3} - \dots;$$

whence $\log 11$ is found to be $1\cdot 0413927$.

NOTE. We may also find $\log 7$ quickly as follows:

By putting $n=50$ in series (2), we get $\log 50 - \log 49$; thus

$$2 - \log 2 - 2 \log 7 = \frac{\mu}{50} + \frac{\mu}{2 \cdot 50^2} + \frac{\mu}{3 \cdot 50^3} + \dots$$

555. We shall now give some further examples on the logarithmic series. In all cases it is assumed that the symbols are such as to make the expansions legitimate.

EXAMPLE 1. To find $\log 10008$ to seven decimal places.

Since $\log 10008 = \log (10^4 \times 1.0008) = 4 + \log 1.0008$, it is only necessary to find $\log 1.0008$ by putting $x = .0008$ in the series

$$\log_{10}(1+x) = \mu \left\{ x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \right\}.$$

Thus to seven places of decimals,

$$\begin{aligned} \log 10008 &= 4 + .43429448 \{ .0008 - \frac{1}{2} (.0008)^2 \} \\ &= 4.0003473. \end{aligned}$$

EXAMPLE 2. If α, β are the roots of the equation $ax^2 - bx + c = 0$, shew that

$$\log(a - bx + cx^2) = \log a - (\alpha + \beta)x - \frac{\alpha^2 + \beta^2}{2}x^2 - \frac{\alpha^3 + \beta^3}{3}x^3 - \dots$$

By the Theory of Quadratics, we have

$$\alpha + \beta = \frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

Now

$$\begin{aligned} a - bx - cx^2 &= a \left(1 - \frac{b}{a}x + \frac{c}{a}x^2 \right) \\ &= a \{ 1 - (\alpha + \beta)x + \alpha\beta x^2 \} \\ &= a(1 - \alpha x)(1 - \beta x). \end{aligned}$$

$$\begin{aligned} \therefore \log(a - bx + cx^2) &= \log a + \log(1 - \alpha x) + \log(1 - \beta x) \\ &= \log a - \left(\alpha x + \frac{\alpha^2 x^2}{2} + \frac{\alpha^3 x^3}{3} + \dots \right) - \left(\beta x + \frac{\beta^2 x^2}{2} + \frac{\beta^3 x^3}{3} + \dots \right) \\ &= \log a - (\alpha + \beta)x - \frac{\alpha^2 + \beta^2}{2}x^2 - \frac{\alpha^3 + \beta^3}{3}x^3 - \dots \end{aligned}$$

EXAMPLE 3. If $\log_e \frac{1}{1+x+x^2+x^3}$ is expanded in ascending powers of x , shew that the coefficient of x^n is $\frac{3}{n}$ if n is a multiple of 4, and $-\frac{1}{n}$ if n is not a multiple of 4.

$$\log_e \frac{1}{1+x+x^2+x^3} = \log_e \frac{1-x}{1-x^4} = \log_e(1-x) - \log_e(1-x^4).$$

(i) If n is not a multiple of 4, the term involving x^n comes only from $\log_e(1-x)$;

$$\therefore \text{the required coefficient} = -\frac{1}{n}.$$

(ii) If n is a multiple of 4, put $n = 4m$;

$$\text{then the required coefficient} = -\frac{1}{4m} + \frac{1}{m} = \frac{3}{4m} = \frac{3}{n}.$$

EXAMPLES XLIII

1. Find the coefficient of x^r in the expansions of

$$(i) \frac{1-x}{e^x}; \quad (ii) \frac{ax+b}{e^x}.$$

2. Find the coefficient of x^n in the series

$$1 + \frac{a+bx}{1} + \frac{(a+bx)^2}{2} + \dots + \frac{(a+bx)^r}{r} + \dots$$

3. Shew that

$$(i) e^{-2} = 1 - \frac{2^3}{3} + \frac{2^4}{4} - \frac{2^5}{5} + \dots; \quad (ii) \frac{e^2-1}{2e} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$$

4. Find the coefficient of x^r in the expansion of $\frac{1-x-x^2}{e^x}$.

5. Shew that $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots = \frac{1}{2}(e + e^{-1})$.

6. Shew that $e^{-1} = 2 \left(\frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \dots \right)$.

7. Prove that

$$\left\{ 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \right\} \left\{ 1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \dots \right\} = 1.$$

Find the sum of the following infinite series :

$$8. \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots$$

$$9. \frac{1^2}{2} + \frac{2^2}{3} + \frac{3^2}{4} + \frac{4^2}{5} + \dots$$

$$10. 1 + \frac{1+2}{1 \cdot 2} + \frac{1+2+3}{1 \cdot 2 \cdot 3} + \frac{1+2+3+4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots$$

$$11. a^2 - b^2 + \frac{1}{2}(a^4 - b^4) + \frac{1}{3}(a^6 - b^6) + \dots$$

12. Expand $\log \sqrt{1+x}$ in ascending powers of x .

13. Shew that $\log_{10} \left(\frac{1}{1-x} \right) = \frac{1}{\log_e 10} \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \right)$.

14. If $y = -x + \frac{x^2}{2} - \frac{x^3}{3} + \dots$, and y is less than 1, express x in a series of ascending powers of y .

15. Shew that

$$\log_e \frac{m}{n} = 2 \left\{ \frac{m-n}{m+n} + \frac{1}{3} \left(\frac{m-n}{m+n} \right)^3 + \frac{1}{5} \left(\frac{m-n}{m+n} \right)^5 + \dots \right\}.$$

By putting $m=2$, and $n=1$, calculate the value of $\log_e 2$ to five decimal places.

16. Expand $\log_e (1+x-2x^2)$ in ascending powers of x to four terms, and find the general term.

17. Find the general term of the expansion of $\log_e (1+2x-8x^2)$. Thence write down the first four terms of the series.

18. Prove that

$$\log_e \frac{1+x}{1-3x} = 4x + 4x^2 + \frac{28}{3}x^3 + 20x^4 + \dots,$$

and find the general term of the series.

19. Prove that the coefficient of x^n in the expansion of $\log_e (1+x+x^2)$ is $-\frac{2}{n}$ or $\frac{1}{n}$ according as n is or is not a multiple of 3.

20. If α and β are the roots of the equation $x^2 - px + q = 0$, shew that

$$\log_e (1+px+qx^2) = (\alpha+\beta)x - \frac{\alpha^2+\beta^2}{2}x^2 + \frac{\alpha^3+\beta^3}{3}x^3 - \dots$$

Deduce the expansion of $\log_e (1+3x+2x^2)$.

21. Prove that

$$\log_e (n+1) - \log_e (n-1) = 2 \left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \dots \right).$$

Thence find the Napierian logarithm of $\frac{1001}{999}$ correct to ten decimal places.

22. If $x < 1$, find the sum of the infinite series

$$\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \frac{4}{5}x^5 + \dots$$

23. Prove that

$$\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \dots = \frac{1}{17} + \frac{1}{19} + \frac{1}{3} \left(\frac{1}{17^3} + \frac{1}{19^3} \right) + \dots$$

24. Assuming $\log 2 = .3010300$, $\log 3 = .4771213$, and $\frac{1}{\log_e 10} = .43429448$, calculate the values of $\log 13$ and $\log 17$ to five places of decimals.

25. Prove that

$$\log a + \log \frac{a^2}{b} + \log \frac{a^3}{b^2} + \dots \text{ to } n \text{ terms} = \frac{n}{2} (\log a^{n+1} - \log b^{n-1}).$$

If $a=25$, $b=2$, and $n=100$, calculate the sum of the series to the nearest integer.

CHAPTER XLIV

COMPOUND INTEREST AND ANNUITIES

556. PROBLEMS connected with Compound Interest and Annuities give useful practice in logarithmic work. We shall here prove and illustrate the necessary formulæ for the solution of such problems, using the terms and phraseology of the subject in their ordinary arithmetical sense.

One difference of usage should be noted: instead of taking as the rate of interest the interest on £100 for one year, it will be found more convenient to take the interest on £1 for one year. Thus our formulæ will involve a symbol for the rate *per pound* instead of the usual “rate *per cent.*”

557. *To find the interest and amount in n years of a given sum at compound interest.*

Let P denote the principal, r the interest on £1 for one year, M the amount, all expressed in pounds.

Let $\text{£}R$ denote the *amount* of £1 in one year; then $R = 1 + r$.

The amount of P at the end of the first year is PR ; and, since this is the principal for the second year, the amount at the end of the second year is $PR \times R$ or PR^2 . Similarly the amount at the end of the third year is $PR^2 \times R$ or PR^3 , and so on; hence the amount in n years is PR^n ;

that is,
$$M = PR^n = P(1 + r)^n.$$

Also the interest $= M - P = P(R^n - 1)$.

By the aid of logarithms any of the four quantities involved in the formula $M = PR^n$ may be found when the other three are known.

558. *To find the present value and discount of a given sum due in a given time, allowing compound interest.*

Let P be the given sum, V the present value, D the discount, R the amount of £1 for one year, n the number of years.

Since V is the sum which, put out to interest at the present time, will in n years amount to P , we have

$$P = VR^n;$$

$$\therefore V = PR^{-n}.$$

Also

$$D = P - V = P(1 - R^{-n}).$$

559. If interest is paid more than once a year, and each instalment of interest, as it becomes due, is added to the principal, the formula for M requires modification.

Thus if interest is paid q times a year, the interest of £1 for each period is $\frac{r}{q}$, and therefore in q years, or nq periods,

$$M = P \left(1 + \frac{r}{q} \right)^{nq}.$$

In this case the interest is said to be “converted into principal” q times a year.

EXAMPLE 1. Find to the nearest pound the amount of £100 in 15 years, allowing compound interest at 4%, convertible half-yearly.

Here $R = 1 + \frac{1}{2} \cdot \frac{4}{100} = 1.02$, and the number of payments is 30.

Hence $M = 100(1.02)^{30}$;

$$\therefore \log M = 2 + 30 \log 1.02 = 2.258$$

$$= \log 181.1, \text{ from the Tables.}$$

$\therefore$ the required amount = £181, to the nearest pound.

With four-figure logarithms a more accurate result cannot be obtained. In some of the examples which follow it will be necessary to use logarithms taken from seven-figure Tables.

EXAMPLE 2. Find in how many years £1130 will amount to £3000 at 5% compound interest.

If n be the number of years, we have $3000 = 1130(1.05)^n$.

$$\therefore \log 3000 = \log 1130 + n \log 1.05,$$

$$3.4771 = 3.0531 + n(.0212),$$

$$\text{that is, } n = \frac{3.4771 - 3.0531}{.0212} = \frac{.4240}{.0212} = 20.$$

Thus the number of years is 20.

EXAMPLE 3. Find the present value of £6000 due in 20 years, allowing compound interest at 4% per annum. Given

$$\log 6 = .7781513, \log 104 = 2.0170333, \log 2.73833 = .4374853.$$

Let £ V denote the present value; then

$$6000 = V(1.04)^{20};$$

$$\therefore 3 + \log 6 = \log V + 20(.0170333);$$

$$\begin{aligned} \text{that is, } \log V &= 3 + .7781513 - .3406660 \\ &= 3.4374853; \end{aligned}$$

$$\text{whence } V = 2738.33.$$

Thus the present value = £2738.33, or £2738, to the nearest pound.

[Examples XLIV. 1–8, page 522, may be taken here.]

Annuities

560. An annuity is a fixed sum, paid under certain stated conditions, at regular intervals of time. Unless it is otherwise stated we shall suppose the payments annual.

If the annuity is payable unconditionally for a fixed term of years it is called an **annuity certain**. If the annuity is to continue for ever it is called a **perpetuity**.

561. *To find the amount of an annuity left unpaid for a given number of years, allowing compound interest.*

Let A be the annuity, R the amount of £1 for one year, n the number of years, M the amount.

At the end of the first year A is due, and the amount of this sum for the remaining $n - 1$ years is AR^{n-1} ; at the end of the second year A is again due, and the amount of this sum in the remaining $n - 2$ years is AR^{n-2} ; and so on.

$$\begin{aligned}\therefore M &= AR^{n-1} + AR^{n-2} + \dots + AR^2 + AR + A \\ &= A(1 + R + R^2 + \dots \text{ to } n \text{ terms}) \\ &= A \frac{R^n - 1}{R - 1}.\end{aligned}$$

562. Part of the business of Life Insurance Companies is to grant annuities, payable over a stated period, in return for a sum of money paid down. This sum is the **purchase price** or **present value** of the annuity.

563. *To find the present value of an annuity to continue for a given number of years, allowing compound interest.*

Let A be the annuity, R the amount of £1 in one year, n the number of years, V the required present value.

The present value of A due in 1 year is AR^{-1} ;
the present value of A due in 2 years is AR^{-2} ;
the present value of A due in 3 years is AR^{-3} ; and so on. [Art. 558.]

Now V is the sum of the present value of these different payments.

$$\begin{aligned}\therefore V &= AR^{-1} + AR^{-2} + AR^{-3} + \dots \text{ to } n \text{ terms} \\ &= AR^{-1} \frac{1 - R^{-n}}{1 - R^{-1}} \\ &= A \frac{1 - R^{-n}}{R - 1}.\end{aligned}$$

COR. Since $R > 1$, R^{-n} becomes indefinitely small when n is infinite.

Hence the present value of a *perpetuity* $= \frac{A}{R - 1} = \frac{A}{r}$.

564. If mA is the present value of an annuity A , the annuity is said to be worth m years' purchase.

In the case of a perpetual annuity $mA = \frac{A}{r}$;

hence $m = \frac{1}{r} = \frac{100}{\text{rate per cent.}}$;

that is, the number of years' purchase of a perpetual annuity is obtained by dividing 100 by the rate per cent.

Irredeemable Stocks, such as some Government Securities, Corporation Stocks, Railway Debentures, are examples of perpetual annuities. In applying the above formula to any given case it must be remembered that the numerator is the *current value of £100 stock*. Thus when $2\frac{1}{2}$ p.c. Consols were quoted at 80 they were worth 32 years' purchase.

565. A freehold estate is an estate which yields a perpetual annuity called the *rent*; and thus the value of the estate is equal to the present value of a perpetuity equal to the rent. Hence if we know the number of years' purchase of an estate, we can obtain the rate per cent. at which interest is reckoned by dividing 100 by the number of years' purchase.

EXAMPLE 1. Find the amount of an annuity of £100 in 15 years, allowing compound interest at 4 per cent. per annum. Given

$$\log 1.04 = .01703, \text{ and } \log 1.80085 = 5.25545.$$

We have $M = 100 \frac{(1.04)^{15} - 1}{.04} = 2500 \{(1.04)^{15} - 1\}.$

Now $\log (1.04)^{15} = 15 \times .01703 = .25545 = \log 1.80075$;

whence $(1.04)^{15} = 1.80075.$

$$\therefore M = 2500 \times .80075 = 2001.875.$$

Thus the required amount = £2001.875, or £2001. 17s. 6d.

EXAMPLE 2. A man borrows £20000 at 5 per cent. compound interest. If the principal and interest are to be paid by 20 equal annual instalments, find approximately the amount of each of these.

Let £ A be the value of each instalment; then £20000 is the present value of an annuity of £ A payable for 20 years.

Hence $20000 = A \frac{1 - (1.05)^{-20}}{.05}$; whence $A\{1 - (1.05)^{-20}\} = 1000.$

Now $\log (1.05)^{-20} = -20(.0212) = -.424 = \bar{1}.576 = \log .3767$;

whence $(1.05)^{-20} = .3767.$

$$\therefore A(1 - .3767) = 1000; \text{ whence } A = \frac{1000}{.6233} = 1604 \text{ nearly.}$$

Thus the value of each instalment is £1604.

566. A deferred annuity, or reversion, is an annuity which does not begin until after the lapse of a certain number of years. When the annuity is deferred for n years, it is said to begin *after* n years, and the first payment is made at the end of $n + 1$ years.

567. To find the present value of a deferred annuity to begin at the end of p years and to continue for n years, allowing compound interest.

Let A be the annuity, R the amount of £1 in one year, V the present value.

The first payment is made at the end of $p + 1$ years.

Hence the present values of the first, second, third, ... payments are respectively

$$\begin{aligned} & AR^{-(p+1)}, \quad AR^{-(p+2)}, \quad AR^{-(p+3)}, \quad \dots \\ \therefore V &= AR^{-(p+1)} + AR^{-(p+2)} + AR^{-(p+3)} + \dots \text{ to } n \text{ terms} \\ &= AR^{-(p+1)} \cdot \frac{1 - R^{-n}}{1 - R^{-1}} \\ &= \frac{AR^{-p}}{R - 1} - \frac{AR^{-p-n}}{R - 1}. \end{aligned}$$

COR. The present value of a *deferred perpetuity* to begin after p years is obtained by making n infinite.

In this case
$$V = \frac{AR^{-p}}{R - 1}.$$

EXAMPLE. The reversion of an estate worth £450 per annum is bought for £5000. Within what time must the buyer take possession so as not to lose by his purchase, supposing interest to be at 5 per cent. ?

Let n be the number of years ; then £5000 is the present value of a perpetuity of £450 deferred for n years.

$$\therefore 5000 = \frac{450(1.05)^{-n}}{.05}; \text{ whence } 5 = 9(1.05)^{-n}.$$

$$\therefore \log 5 = \log 9 - n \log 1.05;$$

that is,
$$n = \frac{\log 9 - \log 5}{\log 1.05} = \frac{.9542 - .6990}{.0212} = \frac{.2552}{.0212} = 12.$$

Thus he must take possession in 12 years.

EXAMPLES XLIV

[Use four-figure Tables unless special logarithms are quoted.]

Find, to the nearest pound, the amount at compound interest of

1. £370 in 25 yrs. at 4%.
2. £450 in 20 yrs. at $2\frac{1}{2}\%$.
3. What sum will amount to £3000 in 15 years at $3\frac{1}{2}\%$?
4. In what time will £P become £100P at $5\frac{1}{2}\%$?

5. At 5% for $6\frac{1}{2}$ years, prove the formula $M = P \times (1.05)^6 \times 1.025$. Hence find the present value of £3000 due in $6\frac{1}{2}$ years at 5%.

6. At what rate per cent. will £50 become £5000 in 50 years ?

7. Find, to the nearest pound, the amount at compound interest of £6000 in 10 years at 5% per annum paid quarterly.

Given $\log 1.0125 = .0054$.

8. If the rate of interest is such that a sum of money doubles itself in 10 years, shew that £1 will amount to £1000 in about 100 years.

9. Find the amount of an annuity of £250 left unpaid for 12 years at 4%.

Given $\log 1.04 = .01703$, $\log 1.6009 = .20436$.

10. Find the present value of an annuity of £900 to continue for 20 years at $4\frac{1}{2}$ %.

Given $\log 1.045 = .01912$, $\log 41458 = 4.6176$.

11. A Corporation borrows £5000 to be repaid with interest at 3% in 10 equal annual instalments. What sum (to the nearest pound) must be repaid each year ?

12. If at the beginning of each year a man invested £50 at 4% compound interest, find to the nearest shilling what his savings amounted to at the end of 20 years.

Given $\log 1.04 = .0170333$, $\log 2.19112 = .3406660$.

13. Calculate at 3% the purchase price of an annuity of £150 to continue for 20 years, the first payment to be made one year from the date of purchase.

Given $\log 1.03 = .0128372$, $\log 5.53677 = .7432560$.

14. A freehold estate worth £180 a year is sold for £4000; find the rate of interest.

15. How many years' purchase is a freehold estate worth at $6\frac{1}{4}$ %.

16. If a perpetuity is worth 25 years' purchase, find at the same rate of interest the amount of an annuity of £500 to continue for 2 years.

17. The reversion after 6 years of a freehold estate is bought for £20000; at what rent should it be let so that the owner may receive 5% on the purchase money ?

Given $\log 1.05 = .0211893$, $\log 1.340096 = .1271358$.

18. What is the present value of a perpetual annuity of £10 payable at the end of the first year, £20 at the end of the second, £30 at the end of the third, and so on, increasing £10 each year; interest being taken at 5% per annum ?

CHAPTER XLV

SCALES OF NOTATION

568. THE common or denary scale of notation is that in which ordinary arithmetical numbers are expressed by means of multiples of powers of 10 ; for instance

$$\begin{aligned} 235 &= 2 \times 10^2 + 3 \times 10 + 5, \\ 9041 &= 9 \times 10^3 + 0 \times 10^2 + 4 \times 10 + 1. \end{aligned}$$

In this system ten is said to be the **radix** of the scale, and the necessary symbols are the ten digits 0, 1, 2, 3, ... 9. Any number other than ten may be taken as the radix of a scale of notation ; thus, if 7 is the radix, a number expressed by 2503 represents

$$2 \times 7^3 + 5 \times 7^2 + 0 \times 7 + 3.$$

In this scale no digit higher than 6 can occur.

More generally, a number in a scale whose radix is r may be expressed as follows :

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0,$$

where $a_0, a_1, a_2, \dots, a_n$ represent the digits in order, beginning with that in the units' place. Each of these digits is a positive integer or zero, and each must be less than r .

It must be remembered that, except in the denary scale, a number expressed by 10 does not stand for *ten*, but for the radix itself.

569. The ordinary operations of Arithmetic may be performed in any scale ; but as the powers of the radix are no longer powers of *ten*, in determining the *carrying figures* we must divide not by ten, but by the radix of the scale we are considering.

EXAMPLE. Find the sum of 3264, 5042, 1465 in the scale of seven, and subtract 4541 from the result.

- | | |
|---|---|
| <p>(i) 3264
5042
1465
<hr/>13134
4541
<hr/>5263</p> | <p>(i) Here 5, 2, 4 make <i>eleven</i>, or 1 <i>seven</i> + 4 ;
set down 4 and carry 1.</p> <p>7, 4, 6 make <i>seventeen</i>, or 2 <i>sevens</i> + 3 ;
set down 3 and carry 2 ; and so on.</p> <p>(ii) After the first step of subtraction, since we cannot take 4 from 3, we add <i>seven</i> ; thus we have to take 4 from ten which leaves 6 ; then 6 from eight, which leaves 2 ; and finally 5 from ten, which leaves 5.</p> |
|---|---|

570. The names *binary*, *ternary*, *quaternary*, *quinary*, *senary*, *septenary*, *octonary*, *nonary*, *denary*, *undenary*, and *duodenary* (or *duodecimal*) are used to denote the scales corresponding to the values *two*, *three*, ... *twelve* of the radix. We shall not consider any scale higher than these. In the undenary and duodenary scales we shall use the symbols *t* and *e* as digits to denote *ten* and *eleven* respectively.

EXAMPLE. Divide 15et20 by 9 in the scale of twelve.

Here $15 = 1 \text{ twelve} + 5 = \text{seventeen} = 1 \times 9 + 8$.

$\therefore$ we set down 1 and carry 8.

$$\begin{array}{r} 9 \overline{) 15et20} \\ 1ee96 \dots 6 \end{array}$$

Also $8 \times \text{twelve} + e = \text{one hundred and seven} = e \times 9 + 8$.

$\therefore$ we set down *e* and carry 8; and so on.

571. To express a given integer *N* in any new scale.

Let *r* be the radix of the new scale, and let $a_0, a_1, a_2, \dots, a_n$ be the required digits by which *N* is to be expressed, beginning with that in the units' place; then

$$N = a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0.$$

We have now to find the values of $a_0, a_1, a_2, \dots, a_n$.

Divide *N* by *r*, then the remainder is a_0 , and the quotient is

$$a_n r^{n-1} + a_{n-1} r^{n-2} + \dots + a_2 r + a_1.$$

If this quotient is divided by *r*, the remainder is a_1 ;
if the next quotient " " " " a_2 ;

and so on, until there is no further quotient divisible by *r*.

Thus the required digits are the remainders found by successive divisions by *the radix of the new scale*.

EXAMPLE 1. Express the denary number 4213 in the scale of nine.

$$\begin{array}{r} 9 \overline{) 4213} \\ 9 \overline{) 468} \dots 1 \\ 9 \overline{) 52} \dots 0 \\ 5 \dots 7 \end{array}$$

Here we divide successively by 9 (the new radix), performing the division in the scale of the given number.

$$\text{Thus } 4213 = 5 \times 9^3 + 7 \times 9^2 + 0 \times 9 + 1.$$

$\therefore$ the required number is 5701.

EXAMPLE 2. Transform 21125 from scale seven to scale eleven.

Here we work in the scale of seven; thus

$$21 = 2 \times \text{seven} + 1 = \text{fifteen} = 1 \times e + 4.$$

$\therefore$ we set down 1 and carry 4.

$$\text{Next } 4 \times 7 + 1 = \text{twenty-nine} = 2 \times e + 7.$$

$\therefore$ we set down 2 and carry 7; and so on.

The successive remainders are *t*, 0, *t*, and the last quotient is 3.

Thus the required number is 3t0t.

[Examples XLV. 1–12, page 530, may be taken here.]

Radix Fractions

572. Fractions may also be expressed in any scale of notation ; thus,
 just as $\cdot 253$ in scale ten denotes $\frac{2}{10} + \frac{5}{10^2} + \frac{3}{10^3}$,
 so $\cdot 253$ in scale 6 denotes $\frac{2}{6} + \frac{5}{6^2} + \frac{3}{6^3}$,
 and $\cdot 253$ in scale r denotes $\frac{2}{r} + \frac{5}{r^2} + \frac{3}{r^3}$.

Fractions thus expressed are called **radix-fractions**. The general type of such fractions in scale r is

$$\frac{b_1}{r} + \frac{b_2}{r^2} + \frac{b_3}{r^3} + \dots ;$$

where $b_1, b_2, b_3, \dots$ are integers, all less than r , of which any one or more may be zero.

573. To express a given radix-fraction F in any new scale.

Let r be the radix of the new scale, and let $b_1, b_2, b_3, \dots$ be the required digits by which F is to be expressed, beginning from the left ; then

$$F = \frac{b_1}{r} + \frac{b_2}{r^2} + \frac{b_3}{r^3} + \dots$$

We have now to find the values of $b_1, b_2, b_3, \dots$.

Multiply both sides of the equation by r ; then

$$rF = b_1 + \frac{b_2}{r} + \frac{b_3}{r^2} + \dots$$

Hence b_1 is equal to the integral part of rF ; and, if we denote the fractional part by F_1 , we have

$$F_1 = \frac{b_2}{r} + \frac{b_3}{r^2} + \dots$$

Multiply again by r ; then b_2 is the integral part of rF_1 . In the same way by successive multiplications by r , each of the digits may be found, and the fraction expressed in the new scale.

EXAMPLE 1. Express $\frac{7}{8}$ (scale ten) as a radix-fraction in scale six.

$$\frac{7}{8} \times 6 = \frac{7 \times 3}{4} = 5 + \frac{1}{4} ;$$

$$\frac{1}{4} \times 6 = \frac{1 \times 3}{2} = 1 + \frac{1}{2} ;$$

$$\frac{1}{2} \times 6 = 3.$$

Here, as in Art. 571, we multiply successively by the radix of the new scale, performing the work in the scale of the given fraction.

$$\therefore \text{ the required fraction } = \frac{5}{6} + \frac{1}{6^2} + \frac{3}{6^3} = \cdot 513.$$

EXAMPLE 2. Transform 606·7 from scale eight to scale five.

Here we must treat the integral and fractional parts separately.

$$\begin{array}{r} 5 \overline{) 606} \quad .7 \\ 5 \overline{) 116} \dots 0 \\ 5 \overline{) 17} \dots 3 \\ 3 \dots 0 \end{array} \quad \begin{array}{r} .7 \\ 5 \\ 4.3 \\ 5 \\ 1.7 \end{array}$$

Here we divide or multiply by the new radix, performing the work in the scale of the given number.

The digits of the radix-fraction recur; hence the required number is 3030·41.

Some Properties of Numbers.

574. In any scale of notation of which the radix is r , the sum of the digits of any whole number when divided by $r-1$ will leave the same remainder as the whole number when divided by $r-1$.

Let N denote the number, $a_0, a_1, a_2, \dots, a_n$, the digits beginning with that in the units' place, and S the sum of the digits;

then
$$N = a_0 + a_1 r + a_2 r^2 + \dots + a_{n-1} r^{n-1} + a_n r^n;$$

$$S = a_0 + a_1 + a_2 + \dots + a_{n-1} + a_n.$$

$$\therefore N - S = a_1(r-1) + a_2(r^2-1) + \dots + a_{n-1}(r^{n-1}-1) + a_n(r^n-1).$$

Now every term on the right is divisible by $r-1$;

$$\therefore \frac{N-S}{r-1} = \text{an integer}; \text{ that is, } \frac{N}{r-1} = I + \frac{S}{r-1},$$

where I is some integer; which proves the proposition.

Hence a number in scale r is divisible by $r-1$ when the sum of its digits is divisible by $r-1$.

575. A denary number divided by 9 leaves the same remainder as the sum of its digits divided by 9. The rule known as "casting out the nines" for testing the accuracy of multiplication is founded on this property. The rule may be explained as follows:

Let two numbers be represented by $9a+b$ and $9c+d$, and their product by P ; then $P = 81ac + 9bc + 9ad + bd$.

Hence $P/9$ has the same remainder as $bd/9$; and therefore the sum of the digits of P , when divided by 9, gives the same remainder as the sum of the digits of bd , when divided by 9. If on trial this should not be the case, the multiplication must have been incorrectly performed. In practice b and d are readily found from the sums of the digits of the two numbers.

Thus, to test the accuracy of $4758 \times 827 = 3935866$:

$4+7=11$; cast out 9, and 2 is left. $2+5+8=15$; cast out 9, and the remainder is 6. Similarly the remainder from 827 is 8. The remainder from the product 6×8 is 3. Again the remainder from 3935866 is 4. Thus the result is not correct.

576. Any number of n digits can be expressed by the formula

$$N = a_{n-1}r^{n-1} + a_{n-2}r^{n-2} + \dots + a_2r^2 + a_1r + a_0.$$

The smallest value N can have is when $a_{n-1} = 1$ and all the other digits are zero. In this case $N = r^{n-1}$.

The greatest value of N is obtained by making each digit as large as possible, that is, by putting

$$a_{n-1} = a_{n-2} = \dots = a_2 = a_1 = a_0 = r - 1.$$

In this case $N = (r - 1)(r^{n-1} + r^{n-2} + \dots + r^2 + r + 1) = r^n - 1$.

Thus in scale r a number of n digits cannot be less than r^{n-1} , nor greater than $r^n - 1$; that is, it must be less than r^n .

EXAMPLE. If N is a denary number of n digits, how many digits are there in the square of N ?

N^2 is less than $10^n \times 10^n$, or 10^{2n} ; also it is not less than $10^{n-1} \times 10^{n-1}$, or 10^{2n-2} . Now 10^{2n} (expressed by 1 followed by $2n$ ciphers) is the smallest number with $2n + 1$ digits. Similarly, 10^{2n-2} is the smallest number with $2n - 1$ digits.

$\therefore N^2$ cannot have more than $2n$ digits, nor fewer than $2n - 1$.

577. If the square root of a number consists of $2n + 1$ figures, when the first $n + 1$ of these have been obtained by the ordinary method, the remaining n may be obtained by division.

Let N denote the given number; a the part of the square root already found, that is the first $n + 1$ digits found by the common rule, with n ciphers annexed; x the remaining part of the root.

Then

$$\sqrt{N} = a + x;$$

$$\therefore N = a^2 + 2ax + x^2;$$

$$\therefore \frac{N - a^2}{2a} = x + \frac{x^2}{2a} \dots \dots \dots (1)$$

Now $N - a^2$ is the remainder after $n + 1$ digits of the root, represented by a , have been found; and $2a$ is the divisor at the same stage of the work. We see from (1) that $N - a^2$ divided by $2a$ gives x , the rest of the quotient required, increased by $\frac{x^2}{2a}$.

We shall shew that $\frac{x^2}{2a}$ is a proper fraction, so that by neglecting the remainder arising from the division, we obtain x , the rest of the root.

For x contains n digits, and therefore x^2 contains $2n$ digits at most; also a is a number of $2n + 1$ digits (the last n of which are ciphers) and thus $2a$ contains $2n + 1$ digits at least; and therefore $\frac{x^2}{2a}$ is a proper fraction.

EXAMPLE. Find the first 7 figures of the square root of 2.

(i)		(ii)	
	2.00,00 ... (1.414213 ...)		2.00,00 ... (1.414
24	$\begin{array}{r} 1 \\ \hline 100 \\ 96 \end{array}$	24	$\begin{array}{r} 100 \\ \hline 400 \end{array}$
281	$\begin{array}{r} 400 \\ 281 \end{array}$	281	$\begin{array}{r} 400 \\ \hline 11900 \end{array}$
2824	$\begin{array}{r} 11900 \\ 11296 \end{array}$	2824	$\begin{array}{r} 11900 \\ \hline 604(213 \end{array}$
2828:2	$\begin{array}{r} 604 \mid 00^* \\ 565 \mid 64 \end{array}$	2828	$\begin{array}{r} 38 \\ \hline 10 \\ \hline 2 \end{array}$
2828:41	$\begin{array}{r} 38 \mid 3600 \\ 28 \mid 2841 \end{array}$		
2828:423	$\begin{array}{r} 10 \mid 075900 \\ 8 \mid 485269 \end{array}$		
	$\begin{array}{r} 1 \mid 590631 \end{array}$		

Thus the required square root = 1.414213 ...

In (i) the work is given in full ; in (ii) the work is contracted.

At the stage marked * *four* digits of the root have been obtained, and the 'trial divisor' consists of *four* digits, viz. 2828. The remainder at this stage is 604, and if instead of bringing down a new period, and appending a new digit to the divisor, we divide 604 by 282(8), cutting off the last digit and using the contracted method, we can obtain *three* new digits of the root, as shewn in (ii), and it is clear that the work is merely the shortened form of that shewn to the left of the vertical line in (i).

578. We give two more examples.

EXAMPLE 1. Shew that 1.44 is a square number in any scale whose radix is greater than 4.

Let r be the radix ; then $1.44 = 1 + \frac{4}{r} + \frac{4}{r^2} = \left(1 + \frac{2}{r}\right)^2$.

Thus the given number is the square of 1.2.

EXAMPLE 2. Express the senary radix-fraction $.50\dot{3}$ as a vulgar fraction in the same scale.

$$\begin{aligned}
 .50\dot{3} &= \frac{5}{6} + \frac{0}{6^2} + \left(\frac{3}{6^3} + \frac{3}{6^4} + \frac{3}{6^5} + \dots \text{to inf.} \right), \text{ in scale ten,} \\
 &= \frac{5}{6} + \frac{3}{6^3} \cdot \frac{1}{1 - \frac{1}{6}} = \frac{153}{180} = \frac{17}{20}, \text{ in scale ten.}
 \end{aligned}$$

Now $17 = 25$ in scale six, } $\therefore$ the required fraction = $\frac{25}{32}$.
 and $20 = 32$,, ,, }

EXAMPLES XLV

Find the value of

1. $2341 + 1234 + 3412 + 4123$ in the scale of five.
2. $437813 + 306218 + 534623$ in the scale of nine.
3. $et06 + 795t + 856e + 3e7t$ in the duodenary scale.
4. $623005 - 341654$ in the septenary scale.
5. (i) 31044×4302 ; (ii) $(3024)^2$ in the quinary scale.
6. Divide 22653 by 26, and 6435 by 222 in the scale of seven.
7. Find the square root of 222521 in the scale of six, and of 14320241 in the scale of five.
8. Express the denary numbers 4532, 860 in the senary scale, and find their product in that scale.
9. Express the septenary numbers 3625, 203116 in scale ten.
10. Transform 54321 from scale six to scale seven.
11. Transform $112t3$ from the duodenary to the septenary scale.
12. Express the quinary number 30014 in powers of twelve.
13. Express the denary fraction $\frac{5}{16}$ in the nonary scale.
14. Express the decimal $\cdot 1375$ as a radix-fraction (i) in the quaternary, (ii) in the octonary scale.
15. Express the denary number $42\cdot 28$ in the quinary scale.
16. Transform $\cdot 20213$ from scale six to scale eight.
17. Transform $20\cdot 73$ from the nonary to the ternary scale.
18. The radix-fraction $\cdot 20\dot{2}$ is in the quinary scale; express it as a vulgar fraction (i) in the denary, (ii) in the quaternary scale.
19. Express in the scale of eleven the greatest and least numbers that can be formed with 4 digits in the scale of seven.
20. In what scale is a hundred denoted by 400? And in what scale is 647 the square of 25?
21. If 432, 565, 708 are in A.P., find the radix of the scale.
22. In what scale are the radix-fractions $\cdot 16$, $\cdot 20$, $\cdot 28$ in G.P.?
23. Divide $264\cdot 734$ by $3t\cdot 08$ in the scale of twelve.
24. Shew how to weigh 227 lbs. using single weights of the series 1 lb., 2 lbs., 4 lbs., 8 lbs., 16 lbs., ...
25. Express the senary radix-fraction $\cdot 31\dot{5}$ as a denary vulgar fraction.
26. Shew that in any scale greater than three $1\cdot 331$ is a perfect cube.
27. N and N' are two numbers expressed with the same digits but in different order. Shew that $N - N'$ is divisible by $r - 1$.
28. If N is a number in the scale of r , and D is the difference between the sums of the digits in the odd and even places; then $N - D$ or $N + D$ is a multiple of $r + 1$.

EASY INEQUALITIES

Some easy cases of Inequalities have already been given in connection with Ratio and the Progressions. [See Arts. 416, 419, 488.]

If $a - b$ is *positive*, a is said to be algebraically *greater* than b .

The sign $>$ is used for the words “is *greater* than.”

Thus $4 > -5$ because $4 - (-5)$, or $4 + 5$ is positive,

In accordance with these definitions zero must be regarded as greater than any negative quantity.

Throughout this chapter we shall suppose that all the symbols denote real positive quantities unless the contrary is explicitly stated.

(i) $a + x > b + x$; (ii) $a - x > b - x$;

(iii) $ax > bx$; (iv) $\frac{a}{x} > \frac{b}{x}$;

582. If $a - x > b$,
by adding x to each side,

$$a > b + x;$$

583. If $a > b$, then evidently $b < a$;
that is, *if the sides of an inequality are transposed, the sign of inequality must be reversed.*

584. If $a > b$, then $a - b$ is positive and $b - a$ is negative; that is, $-a - (-b)$ is negative, and therefore $-a < -b$; hence, if the signs of all the terms of an inequality are changed, the sign of inequality must be reversed.

585. If $a > b$, then $-a < -b$, and therefore $-ax < -bx$; that is,

$$a(-x) < b(-x);$$

hence, if the sides of an inequality are multiplied by the same negative quantity, the sign of inequality must be reversed.

586. If a and b are any real positive quantities $a^2 + b^2 \geq 2ab$.

Since $(a - b)^2$ is always positive, or zero, $a^2 - 2ab + b^2 \geq 0$;

$$\therefore a^2 + b^2 \geq 2ab.$$

Thus, unless a and b are equal, $a^2 + b^2 > 2ab$. Similarly, unless x and y are equal, $x + y > 2\sqrt{xy}$.

A large number of inequalities depend upon these simple results.

EXAMPLE 1. If a, b, c denote positive quantities, prove that

$$(i) \ a^2 + b^2 + c^2 \geq bc + ca + ab; \quad (ii) \ a^3 + b^3 \geq a^2b + ab^2.$$

(i) We have $a^2 + b^2 \geq 2ab$, $b^2 + c^2 \geq 2bc$, $c^2 + a^2 \geq 2ca$.

Adding these results, on dividing by 2, we have

$$a^2 + b^2 + c^2 \geq bc + ca + ab.$$

(ii) Since $a^2 + b^2 \geq 2ab$, we have $a^2 - ab + b^2 \geq ab$;

$$\therefore (a^2 - ab + b^2)(a + b) \geq ab(a + b); \text{ that is, } a^3 + b^3 \geq a^2b + ab^2.$$

EXAMPLE 2. Shew that $a^4 + b^4 > a^3b + ab^3$ unless $a = b$.

The inequality holds if $a^4 + b^4 - a^3b - ab^3$ is positive.

Now $a^4 + b^4 - a^3b - ab^3 = (a^3 - b^3)(a - b) = (a - b)^2(a^2 + ab + b^2)$, and each of these factors is positive.

EXAMPLE 3. If x may have any real value, find which is the greater, $x^3 + 16x$ or $7x^2 + 10$.

By the Remainder Theorem, $x^3 + 16x - (7x^2 + 10)$ has a factor $x - 1$.

$$\begin{aligned} \text{Hence we find } x^3 - 7x^2 + 16x - 10 &= (x - 1)(x^2 - 6x + 10) \\ &= (x - 1)\{(x - 3)^2 + 1\}. \end{aligned}$$

The second factor is always positive; hence $x^3 + 16x$ is greater or less than $7x^2 + 10$ according as x is greater or less than 1.

EXAMPLE 4. To find the maximum value of the product of two quantities whose sum is given.

Let a and b be the two quantities; then $4ab = (a + b)^2 - (a - b)^2$.

But since $a + b$ is constant, we see that the product ab will be greatest when $(a - b)^2$ is zero. Thus the value of the product is greatest when the two quantities are equal.

EXAMPLES XLVI

[In the following examples the student should note each case in which an inequality reduces to an equality for special values of the symbols.]

1. Prove that $(a+b)(b+c)(c+a) \geq 8abc$.

2. Prove that the sum of a real positive quantity and its reciprocal is never less than 2.

3. Prove that $(ab+cd)(ac+bd) \geq 4abcd$.

4. Shew that if $a > b$ and $x > y$, it does not necessarily follow that $ax > by$, if some of the symbols may denote negative quantities.

5. Prove the inequalities :

$$(i) \frac{a}{b^2} + \frac{b}{a^2} > \frac{1}{a} + \frac{1}{b}; \quad (ii) m^2 + \frac{1}{m^2} > m + \frac{1}{m}.$$

6. If $p^2 + q^2 = r^2 + s^2 = 1$, prove that $pr + qs \leq 1$.

7. Prove that $(a+b+c)^2 > 3(bc+ca+ab)$.

8. Prove that $a^3 - 3b^3 > 3a^2b - 5ab^2$, if $a > b$.

9. For what values of x , positive or negative, will $x^4 + x^3 + 2x$ be greater than 4?

10. Shew that $\frac{2x-1}{x^2+2} < \frac{1}{2}$, for real values of x .

11. If $a > b$, shew that $(\sqrt{a} + \sqrt{b})^2 > 4b$ and $< 4a$.

Hence shew that the difference between $\frac{1}{2}(a+b)$ and $\sqrt{ab}$ is less than $\frac{1}{8}(a-b)^2/b$ and greater than $\frac{1}{8}(a-b)^2/a$.

12. If the product of two quantities is constant, shew that their sum is increased by increasing their difference.

13. Under what conditions is $a^3 + b^3 + c^3 > 3abc$?

14. Shew that $6abc \leq \Sigma bc(b+c)$.

15. Shew that $\frac{x+1}{x^2+3}$ lies between $-\frac{1}{6}$ and $\frac{1}{2}$, for real values of x .

16. If $a:b=c:d$, shew that $a+d > b+c$ provided that $a > b$ and $a > c$.

17. If a, b, c, d are unequal quantities, prove that

$$a^2 + b^2 + c^2 + d^2 > 4\sqrt{abcd}.$$

18. If p, q, r, s are positive and arranged in order of magnitude, prove that if $q+r=p+s$, then $qr > ps$.

CHAPTER XLVII

MISCELLANEOUS EQUATIONS

587. IN previous chapters dealing with equations all the principal methods of solution have been explained and illustrated in the text. We shall now give some further examples, in most of which solution is effected by some special artifice.

Equations Involving One Unknown

EXAMPLE 1. *Solve the equation*

$$\frac{x - bc}{b + c} + \frac{x - ca}{c + a} + \frac{x - ab}{a + b} = a + b + c.$$

The equation may be written

$$\left(\frac{x - bc}{b + c} - a \right) + \left(\frac{x - ca}{c + a} - b \right) + \left(\frac{x - ab}{a + b} - c \right) = 0,$$

or
$$\frac{x - (bc + ca + ab)}{b + c} + \frac{x - (bc + ca + ab)}{c + a} + \frac{x - (bc + ca + ab)}{a + b} = 0;$$

$$\therefore \{x - (bc + ca + ab)\} \left\{ \frac{1}{b + c} + \frac{1}{c + a} + \frac{1}{a + b} \right\} = 0.$$

Since the second factor is not zero, we must have

$$x - (bc + ca + ab) = 0, \quad \text{or} \quad x = bc + ca + ab.$$

EXAMPLE 2. *Solve the equation* $\left(\frac{2x + p + r}{2x + q + r} \right)^2 = \frac{x + p}{x + q}.$

The equation may be written

$$\begin{aligned} \left(1 + \frac{p - q}{2x + q + r} \right)^2 &= 1 + \frac{p - q}{x + q}; \\ \therefore \frac{2(p - q)}{2x + q + r} + \frac{(p - q)^2}{(2x + q + r)^2} &= \frac{p - q}{x + q}. \end{aligned}$$

Removing the factor $p - q$, and transposing, we have

$$\begin{aligned} \frac{p - q}{(2x + q + r)^2} &= \frac{1}{x + q} - \frac{2}{2x + q + r} \\ &= \frac{r - q}{(x + q)(2x + q + r)}; \end{aligned}$$

whence

$$(p - q)(x + q) = (r - q)(2x + q + r),$$

or

$$x\{p - q - 2(r - q)\} = r^2 - q^2 - q(p - q);$$

that is,

$$x(p + q - 2r) = r^2 - pq;$$

$$\therefore x = \frac{r^2 - pq}{p + q - 2r}.$$

EXAMPLE 3. Solve the equation

$$\frac{ax+b}{cx+b} + \frac{bx+a}{cx+a} = \frac{(a+b)(x+2)}{cx+a+b}.$$

The right-hand side = $\frac{(ax+b) + (bx+a) + (a+b)}{cx+a+b};$

$$\therefore (ax+b) \left\{ \frac{1}{cx+b} - \frac{1}{cx+a+b} \right\} + (bx+a) \left\{ \frac{1}{cx+a} - \frac{1}{cx+a+b} \right\} = \frac{a+b}{cx+a+b},$$

or
$$\frac{a(ax+b)}{(cx+b)(cx+a+b)} + \frac{b(bx+a)}{(cx+a)(cx+a+b)} = \frac{a+b}{cx+a+b}.$$

By removing $cx+a+b$ from the denominators, and transposing, we have

$$\frac{a(ax+b)}{cx+b} - a + \frac{b(bx+a)}{cx+a} - b = 0;$$

that is,
$$\frac{ax(a-c)}{cx+b} + \frac{bx(b-c)}{cx+a} = 0;$$

$$\therefore \text{either } x=0, \text{ or } (a^2-ac)(cx+a) + (b^2-bc)(cx+b) = 0.$$

In the latter case we have

$$x(a^2c - ac^2 + b^2c - bc^2) = -a^3 + a^2c - b^3 + b^2c,$$

or
$$x\{ac(a-c) + bc(b-c)\} = a^2(c-a) + b^2(c-b).$$

Thus
$$x=0, \text{ or } \frac{a^2(c-a) + b^2(c-b)}{ac(a-c) + bc(b-c)}.$$

EXAMPLE 4. Solve the equation $4^{2x+1} + 16 = 65 \cdot 4^x.$

We have
$$4 \cdot 4^{2x} - 65 \cdot 4^x + 16 = 0.$$

By writing y for 4^x , we obtain

$$4y^2 - 65y + 16 = 0, \quad \text{or} \quad (4y-1)(y-16) = 0;$$

whence
$$y = \frac{1}{4}, \quad \text{or} \quad 16.$$

Thus
$$4^x = \frac{1}{4} = 4^{-1}, \quad \text{or} \quad 4^x = 4^2;$$

$$\therefore x = -1, \quad \text{or} \quad 2.$$

EXAMPLE 5. Solve the equation $\frac{x^2}{3} + \frac{48}{x^2} = 10 \left(\frac{x}{3} - \frac{4}{x} \right).$

Divide each side by 3; then we have

$$\frac{x^2}{9} + \frac{16}{x^2} = \frac{10}{3} \left(\frac{x}{3} - \frac{4}{x} \right).$$

Write y for $\frac{x}{3} - \frac{4}{x}$; then
$$\frac{x^2}{9} + \frac{16}{x^2} = y^2 + \frac{8}{3};$$

$$\therefore y^2 + \frac{8}{3} = \frac{10}{3}y; \quad \text{whence} \quad y = \frac{4}{3}, \quad \text{or} \quad 2;$$

$$\therefore \frac{x}{3} - \frac{4}{x} = \frac{4}{3}, \quad \text{or} \quad \frac{x}{3} - \frac{4}{x} = 2.$$

From these equations we obtain $x = 6, -2, 3 \pm \sqrt{21}.$

EXAMPLE 6. Solve $\frac{\sqrt{x+48} + \sqrt{x}}{\sqrt{x+48} - \sqrt{x}} = \frac{\sqrt{x-4} + \sqrt{3}}{\sqrt{x-4} - \sqrt{3}}$.

By Art. 427 (*Componendo et dividendo*), we have

$$\frac{\sqrt{x+48}}{\sqrt{x}} = \frac{\sqrt{x-4}}{\sqrt{3}}.$$

Squaring and simplifying, we have

$$3x + 144 = x^2 - 4x;$$

whence $x^2 - 7x - 144 = 0$, or $(x - 16)(x + 9) = 0$;

$$\therefore x = 16, \text{ or } -9.$$

Both of these values will be found to satisfy the given equation.

588. Before clearing an equation of radicals any common factor which contains the unknown should be removed by division.

EXAMPLE. Solve the equation

$$\sqrt{x^2 + 4x - 21} + \sqrt{x^2 - x - 6} = \sqrt{6x^2 - 5x - 39}.$$

We have $\sqrt{(x - 3)(x + 7)} + \sqrt{(x - 3)(x + 2)} = \sqrt{(x - 3)(6x + 13)}.$

The factor $x - 3$ can now be removed from every term ;

thus $\sqrt{x + 7} + \sqrt{x + 2} = \sqrt{6x + 13}.$

This equation may now be solved in the usual way as explained on page 349. The solution gives $x = 2$, or $-\frac{5}{3}$.

On trial it will be found that the second of these values does not satisfy the given equation.

By equating the factor $x - 3$ to zero, we have $x = 3$.

Thus finally the required roots are 2 and 3.

589. The artifice used in the following example is sometimes useful.

EXAMPLE. Solve $\sqrt{3x^2 - 7x - 30} - \sqrt{2x^2 - 7x - 5} = x - 5$(1)

Now it is evident that

$$3x^2 - 7x - 30 - (2x^2 - 7x - 5) \equiv x^2 - 25. \text{(2)}$$

Divide each member of (2) by the corresponding member of (1) ;

thus $\sqrt{3x^2 - 7x - 30} + \sqrt{2x^2 - 7x - 5} = x + 5. \text{(3)}$

Now (2) is an *identity* ; that is, it is true for *all* values of x , whereas (1) is satisfied by the values we are seeking ; hence also equation (3) is true for these values.

From (1) and (3) we have, by subtraction,

$$\sqrt{2x^2 - 7x - 5} = 5 ;$$

whence we obtain $x = 6$, or $-\frac{5}{2}$.

Both of these values will be found to satisfy the given equation.

590. Any equation which can be expressed in the form

$$ax^2 + bx + c + p\sqrt{ax^2 + bx + c} = q$$

can be solved by putting $y = \sqrt{ax^2 + bx + c}$.

The resulting equation $y^2 + py = q$ will give two values of y , each of which will give two values of x . We thus have four values of x ultimately. Of these only those which make y positive are solutions of the original equation; the others satisfy the equation

$$ax^2 + bx + c - p\sqrt{ax^2 + bx + c} = q.$$

Numerical examples of this type have already been given on page 350.

591. Reciprocal Equations. When an equation has all its terms brought to one side and arranged in descending order, if the coefficients are the same when read from left to right or right to left, it is known as a **reciprocal equation**, and it is so called because it remains unaltered when x is replaced by its reciprocal $1/x$.

EXAMPLE. Solve $3x^4 - 16x^3 + 26x^2 - 16x + 3 = 0$.

Dividing throughout by x^2 , and rearranging the terms, we have

$$3\left(x^2 + \frac{1}{x^2}\right) - 16\left(x + \frac{1}{x}\right) + 26 = 0.$$

Put $x + \frac{1}{x} = y$; then $x^2 + \frac{1}{x^2} = y^2 - 2$;

$$\therefore 3(y^2 - 2) - 16y + 26 = 0; \text{ whence } y = 2, \text{ or } \frac{10}{3}.$$

Thus we have $x + \frac{1}{x} = 2$, and $x + \frac{1}{x} = \frac{10}{3}$.

These equations give $x = 1, 1, 3, \frac{1}{3}$.

EXAMPLES XLVII. a

Solve the following equations :

1. $\frac{x+a}{x-a} - \frac{x-b}{x+b} = \frac{2(a+b)}{x}$.

2. $\frac{(x+a)(x+b)}{x+a+b} = \frac{(x+c)(x+d)}{x+c+d}$.

3. $\frac{x-a-1}{x-a-2} - \frac{x-a}{x-a-1} = \frac{x-b-1}{x-b-2} - \frac{x-b}{x-b-1}$.

4. $\frac{x+bc}{b-c} + \frac{x+ca}{a-c} + \frac{x-ab}{a+b} = a+b-c$.

5. $\frac{x-2ab}{2a+b} + \frac{x+bc}{b-c} + \frac{x+2ac}{2a-c} = 2a+b-c$.

Solve the equations :

6. $\frac{bc(ax-1)}{a(b+c)} + \frac{ca(bx-1)}{b(c+a)} + \frac{ab(cx-1)}{c(a+b)} = 3.$
7. $\frac{x+p}{x-q} = \left(\frac{2x+p}{2x-q}\right)^2.$
8. $\frac{x+a}{x-b} = \left(\frac{2x+a+c}{2x-b+c}\right)^2.$
9. $\frac{b+c}{x+2a} + \frac{c+a}{x+2b} = \frac{a+b+2c}{x+a+b}.$
10. $\frac{ax+b}{b-x} + \frac{bx+a}{a-x} = \frac{(a+b)(x+2)}{a+b-x}.$
11. $3^{2x+1} + 9 = 28 \cdot 3^x.$
12. $4^x - 9 \cdot 2^x + 8 = 0.$
13. $3\sqrt{x-8} = 3x^{-\frac{1}{2}}.$
14. $6x^{\frac{3}{4}} = 7x^{\frac{1}{4}} - 2x^{-\frac{1}{4}}.$
15. $3^{2x+3} - 55 = 28(3^x - 2).$
16. $6(6^x + 6^{-x}) = 37.$
17. $x^2 + \frac{4}{x^2} = 15\left(\frac{x}{2} + \frac{1}{x}\right) - 17\frac{1}{2}.$
18. $\frac{x^2}{3} + \frac{3}{x^2} = 5\left(\frac{x}{3} - \frac{1}{x}\right).$
19. $\frac{\sqrt{x} + \sqrt{x-15} - \sqrt{x-7}}{\sqrt{x} + \sqrt{x-15} + \sqrt{x-7}} = \frac{1}{4}.$
20. $\frac{p\sqrt{a^2-x^2} + q(a-x)}{p\sqrt{a^2-x^2} - q(a-x)} = \frac{pb+qc}{pb-qc}.$
21. $\sqrt{x^2+14x+33} + \sqrt{x^2-6x-27} = 10\sqrt{x+3}.$
22. $\sqrt{x^2+4x-5} - \sqrt{x^2-12x+11} = \sqrt{x^2-17x+16}.$
23. $\sqrt{2x^2+3x-2} - \sqrt{18x^2+5x-7} + \sqrt{8x^2-2x-1} = 0.$
24. $\sqrt{x^2+6ax+8a^2} + \sqrt{x^2+3ax+2a^2} = 2\sqrt{x^2-4a^2}.$
25. $\sqrt{4x^2-10x+19} - \sqrt{4x^2-10x+3} = 2.$
26. $\sqrt{2x^2-11x+69} + \sqrt{(2x-7)(x-2)} = 11.$
27. $\sqrt{4x^2+8x-28} + \sqrt{3x^2+8x-24} = x+2.$
28. $\sqrt{7x^2-11x+6} + \sqrt{6x^2-11x+15} = 2(x+3).$
29. $\sqrt{3x-24} + \sqrt{x+7} = \sqrt{3x-14} + \sqrt{x-3}.$
30. $8\sqrt{(3x+4)(x+2)} - 3x^2 - 10x + 97 = 0.$
31. $4x^2 + 5x - 2\sqrt{3x^2 - 5x + 2} = x(15 - 2x).$
32. $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0.$
33. $2x^4 - 13x^3 + 24x^2 - 13x + 2 = 0.$
34. $12(x^4 + 1) + 89x^2 = 56x(x^2 + 1).$
35. $6x^4 + 25x^3 + 12x^2 - 25x + 6 = 0.$
36. $(x+9)(x-3)(x-7)(x+5) = 385.$
[Multiply alternate factors together and form a quadratic in $x^2 + 2x$.]
37. $(x+3a)(x-5a)(x^2-16a^2) = 180a^4.$
38. $(1-10x)\sqrt{1+2x} = (1+10x)\sqrt{1-2x}.$

39. $(x-4)^3 + (x-5)^3 = 31\{(x-4)^2 - (x-5)^2\}$.
 40. $4\{(x^2-16)^{\frac{3}{4}} + 8\} = x^2 + 16(x^2-16)^{\frac{1}{4}}$. [Write y^2 for x^2-16 .]
 41. $(a+x)^{\frac{2}{3}} + 4(a-x)^{\frac{2}{3}} = 5(a^2-x^2)^{\frac{1}{3}}$.
 42. $\sqrt{x^2+ax-1} - \sqrt{x^2+bx-1} = \sqrt{a} - \sqrt{b}$.

Equations in two or more Unknowns.

592. The principal methods of solving equations in two unknowns, when either or both of the equations is of higher degree than the first, have been given in Chapter XXVI. Of this class we shall here only give a few additional examples.

EXAMPLE 1. Solve $x^4 + y^4 = 97$,(1)
 $x + y = 5$(2)

First Method.

From (2), we have $(x+y)^4 = 625$,
 or $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 625$(3)

Subtract (1) from (3), and divide the result by 2; then

$2x^3y + 3x^2y^2 + 2xy^3 = 264$,
 that is, $xy(2x^2 + 3xy + 2y^2) = 264$,
 or $xy\{2(x+y)^2 - xy\} = 264$.

Substituting for $x+y$ from (2), we have

$x^2y^2 - 50xy + 264 = 0$, or $(xy-6)(xy-44) = 0$.

Hence we have the two pairs of equations:

$\left. \begin{array}{l} x+y=5, \\ xy=6; \end{array} \right\} \quad \left. \begin{array}{l} x+y=5, \\ xy=44. \end{array} \right\}$

From the first we obtain $x=2, y=3$; $x=3, y=2$.

„ second „ $x = \frac{1}{2}\{5 \pm \sqrt{-151}\}, y = \frac{1}{2}\{5 \mp \sqrt{-151}\}$.

Second Method.

Since $x+y$ is given, assume $x-y=z$; then

$\left. \begin{array}{l} x+y=5, \\ x-y=z; \end{array} \right\} \quad \text{whence } x = \frac{1}{2}(5+z), y = \frac{1}{2}(5-z). \quad \text{.....(4)}$

Substituting in (1), $(5+z)^4 + (5-z)^4 = 97 \times 2^4$.

Hence by expanding $(5+z)^4$ and $(5-z)^4$,

$5^4 + 6 \cdot 5^2z^2 + z^4 = 97 \times 2^3$;

whence $z^4 + 150z^2 - 151 = 0$, or $(z^2-1)(z^2+151) = 0$.

Thus we have $z = \pm 1$, or $z = \pm \sqrt{-151}$; whence from equations (4) we obtain the same values of x and y as before.

EXAMPLE 2. Solve $\frac{x+y}{2x-y} + \frac{x+2y}{x-y} = 8\frac{1}{4}$,(1)

$$2x^2 + 3y^2 = 270. \text{(2)}$$

From (1) $4(x^2 - y^2 + 2x^2 + 3xy - 2y^2) = 33(2x^2 - 3xy + y^2)$;

$$\therefore 18x^2 - 37xy + 15y^2 = 0, \text{ or } (2x - 3y)(9x - 5y) = 0.$$

Hence $2x = 3y$,(3)

or $9x = 5y$(4)

From (3), $\frac{x^2}{9} = \frac{y^2}{4} = \frac{2x^2 + 3y^2}{30} = 9$, by equation (2).

$$\therefore x = \pm 9, \quad y = \pm 6.$$

From (4), $\frac{x^2}{25} = \frac{y^2}{81} = \frac{2x^2 + 3y^2}{293} = \frac{270}{293}$, by equation (2).

$$\therefore x = \pm \frac{15\sqrt{30}}{\sqrt{293}}, \quad y = \pm \frac{27\sqrt{30}}{\sqrt{293}}.$$

[Examples XLVII. b. 1-13, page 542, may be taken here.]

593. Equations of a higher degree than the first, involving more than two unknowns, can only be solved in particular cases. Such equations introduce no new principles, but they often afford scope for considerable skill and ingenuity.

EXAMPLE 1. Solve the equations :

$$x(ax + by + cz) = k, \quad y(ax + by + cz) = l, \quad z(ax + by + cz) = m.$$

Multiply these equations by a, b, c respectively, and add ; then

$$(ax + by + cz)^2 = ak + bl + cm ;$$

$$\therefore ax + by + cz = \pm \sqrt{ak + bl + cm}.$$

$$\therefore x = \pm \frac{k}{\sqrt{ak + bl + cm}}, \quad y = \pm \frac{l}{\sqrt{ak + bl + cm}}, \quad z = \pm \frac{m}{\sqrt{ak + bl + cm}}.$$

EXAMPLE 2. Solve the equations :

$$2x + 5y - 3z = 0, \quad 3x - 9y + z = 0, \quad x^2 - x + 2yz - z = 5.$$

From the first two equations, by cross multiplication (Art. 420),

$$\frac{x}{5-27} = \frac{y}{-9-2} = \frac{z}{-18-15} ;$$

whence

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = k, \text{ suppose ;}$$

then

$$x = 2k, \quad y = k, \quad z = 3k.$$

Substituting these values in the third equation, we get

$$4k^2 - 2k + 6k^2 - 3k = 5,$$

that is,

$$2k^2 - k - 1 = 0, \text{ or } (2k + 1)(k - 1) = 0.$$

From $k = -\frac{1}{2}$, we obtain $x = -1, \quad y = -\frac{1}{2}, \quad z = -\frac{3}{2}.$

„ $k = 1, \quad \text{„} \quad x = 2, \quad y = 1, \quad z = 3.$

EXAMPLE 3. Solve the equations :

$$(y+z)(z+x)=a^2, \quad (z+x)(x+y)=b^2, \quad (x+y)(y+z)=c^2.$$

By multiplying the three equations together, and taking the square root of the result, we obtain

$$(x+y)(y+z)(z+x)=\pm abc.$$

Combining this with each of the given equations, we have

$$x+y=\pm\frac{bc}{a}, \quad y+z=\pm\frac{ca}{b}, \quad z+x=\pm\frac{ab}{c}.$$

From these equations we easily obtain

$$x=\pm\frac{a^2b^2+b^2c^2-c^2a^2}{2abc}, \quad y=\pm\frac{b^2c^2+c^2a^2-a^2b^2}{2abc}, \quad z=\pm\frac{c^2a^2+a^2b^2-b^2c^2}{2abc},$$

EXAMPLE 4. Solve the equations :

$$xy+2x+y=7, \quad yz+3y+2z=12, \quad zx+z+3x=15.$$

By suitable additions to each side of these equations, they may be written

$$xy+(2x+y)+2=9, \quad yz+(3y+2z)+6=18, \quad zx+(z+3x)+3=18,$$

$$\text{or} \quad (x+1)(y+2)=9, \quad (y+2)(z+3)=18, \quad (z+3)(x+1)=18.$$

Solving these equations, as in the last example, we obtain

$$x=2, \quad y=1, \quad z=3;$$

and

$$x=-4, \quad y=-5, \quad z=-9.$$

EXAMPLE 5. Solve (1) $x^2+y^2+z^2=84$, (2) $x+y+z=14$, (3) $xy=z^2$.

From (1) and (2), $(x+y+z)^2-(x^2+y^2+z^2)=112$;

$$\therefore xy+yz+zx=56.$$

Hence from (3), $z^2+yz+zx=56$, or $z(x+y+z)=56$;

whence from (2),

$$z=4.$$

Equations (2) and (3) now become $x+y=10$, $xy=16$.

From these equations we obtain $x=2$, $y=8$; or $x=8$, $y=2$.

Hence finally

$$x=2, \quad y=8, \quad z=4;$$

and

$$x=8, \quad y=2, \quad z=4.$$

EXAMPLE 6. Solve $x^2-yz=1$, $y^2-zx=-5$, $z^2-xy=7$.

Multiply the equations by y , z , x respectively and add; then

$$7x+y-5z=0, \dots\dots\dots(1)$$

Multiply the equations by z , x , y respectively and add; then

$$-5x+7y+1=0. \dots\dots\dots(2)$$

From (1) and (2), $\frac{x}{1+35}=\frac{y}{25-7}=\frac{z}{49+5}$, by cross multiplication,

whence

$$\frac{x}{2}=\frac{y}{1}=\frac{z}{3}=k, \text{ suppose.}$$

Substituting in one of the given equations we get $k=\pm 1$.

$$\therefore x=\pm 2, \quad y=\pm 1, \quad z=\pm 3.$$

EXAMPLES XLVII. b.

Solve the equations :

1. $x^4 + y^4 = 337,$
 $x + y = 7.$
2. $x^4 + y^4 = 272,$
 $x - y = 2.$
3. $x^5 + y^5 = 1023,$
 $x + y = 3.$
4. $\frac{2x + y}{x - 3y} - \frac{x - 3y}{2x + y} = 2\frac{2}{3},$
 $5x + 7y = 19.$
5. $\frac{2x - y}{x + y} + \frac{2y - x}{x + 4y} = 7,$
 $x^2 + y^2 = 29.$
6. $3x^2 + xy = 2x + 6,$
 $y^2 + 3xy = 2y - 3.$
7. $x^2 + xy + 2x + y = 11,$
 $y^2 + xy + 2y + x = 7.$
8. $x^2 - xy + x = 35,$
 $xy - y^2 + y = 15.$
9. $(x - y)^2 = 3 - 2x - 2y,$
 $y(x - y + 1) = x(y - x + 1).$
10. Find the rational roots of
 - (i) $(x + y)(x^3 + y^3) = 19,$
 $x^2 + y^2 = 13;$
 - (ii) $x - y = 2,$
 $(x^2 + y^2)(x^3 - y^3) = 260.$
11. $x + 6y + \frac{x}{y} = 16,$
 $3(x + y) + \frac{x}{y} = 23.$
12. $xy + \frac{1}{xy} + \frac{x}{y} + \frac{y}{x} = 13,$
 $xy - \frac{1}{xy} - \frac{x}{y} + \frac{y}{x} = 12.$
13. $(x + 1)^2 + (x + 1)(y + 2) + (y + 2)^2 = 133,$
 $(x + 1) + \sqrt{(x + 1)(y + 2)} + (y + 2) = 19.$
14. $2x(2x + y + 3z) = 34,$
 $y(2x + y + 3z) = 102,$
 $3z(2x + y + 3z) = 153.$
15. $x(y + z - x) = 39 - 2x^2,$
 $y(x + z - y) = 52 - 2y^2,$
 $z(x + y - z) = 78 - 2z^2.$
16. $3x - 2y - 3z = 0,$
 $x - 10y + 6z = 0,$
 $x^2 + y^2 - z^2 = 116.$
17. $4x - 2y = 7z,$
 $y + z = x,$
 $y^2 + 3z^2 = 4(2x + 1).$
18. $(x - 1)(y + 5) = 14, \quad (y + 5)(z + 8) = 63, \quad (z + 8)(x - 1) = 18.$
19. $xy + x + y = 29, \quad yz + y + z = 23, \quad zx + z + x = 19.$
20. $x^3y^2z = 24, \quad xy^3z^2 = 18, \quad x^2yz^3 = 108.$
21. $x^3y = 2z, \quad y^3z = 9x, \quad xyz = 6.$
22. $x + y + z = 21, \quad x^2 + y^2 + z^2 = 189, \quad y^2 = zx.$
23. $x^2 + y^2 + z^2 = 133, \quad y + z - x = 7, \quad yz = x^2.$
24. $y + z - x = 9, \quad x^2 - y^2 - z^2 = 15, \quad yz = 3.$
25. $x^2 - (y - z)^2 = a^2, \quad y^2 - (z - x)^2 = b^2, \quad z^2 - (x - y)^2 = c^2.$
26. $x^2 - yz = 64, \quad y^2 - zx = 88, \quad z^2 - xy = 4.$
27. $x^2 - y^2 + z^2 = 6, \quad 2yz - zx + 2xy = 13, \quad x - y + z = 2.$

Indeterminate Equations

594. An equation such as $3x + 17y = 130$, in which two unknowns are connected by a single relation, is said to be **indeterminate**. For it is obvious that by giving any value we choose to x , we can obtain a corresponding value of y from the equation. Thus in general the number of solutions of an indeterminate equation is unlimited. If, however, we are restricted to positive integral values of x and y , we may sometimes have a definite number of solutions.

EXAMPLE 1. *A man is to spend £130 in buying sheep at £3 each, and cows at £17 each : how many of each can he buy ?*

Suppose he buys x sheep and y cows ; then

$$3x + 17y = 130.$$

Divide throughout by 3, the smaller of the two coefficients ; then

$$x + 5y + \frac{2y}{3} = 43 + \frac{1}{3};$$

$$\therefore \frac{2y - 1}{3} = 43 - x - 5y.$$

Now since x and y must be integral, we have also

$$\frac{2y - 1}{3} = \text{an integer.}$$

Multiply by a number which will make the coefficient of y differ by unity from a multiple of the denominator ; thus, multiplying by 2, we have

$$\frac{4y - 2}{3} = \text{an integer ;}$$

that is,

$$y + \frac{y - 2}{3} = \text{an integer ;}$$

hence also

$$\frac{y - 2}{3} = \text{an integer}$$

$$= p, \text{ suppose ;}$$

$$\therefore y = 3p + 2. \dots\dots\dots(1)$$

$$\text{Substituting in the original equation we get } x = 32 - 17p. \dots\dots\dots(2)$$

The required values of x and y are now found from (1) and (2) by giving integral values to p .

From (1) it is evident that p cannot be negative, and from (2) we see that p cannot be > 1 .

Thus we have

$$\left. \begin{array}{l} p = 0, 1, \\ x = 32, 15, \\ y = 2, 5, \end{array} \right\}$$

Thus the man may buy 32 sheep and 2 cows, or 15 sheep and 5 cows.

The solution may be verified graphically as follows :

The equation $3x + 17y = 130$ represents a straight line. The positive values of x and y which satisfy the equation are the coordinates of points on that portion of the line which lies in the first quadrant. If the graph is carefully drawn on a sufficiently large scale, it will be found that the only points in the first quadrant which have *integral* coordinates are (32, 2) and (15, 5).

This graphical illustration is left as an exercise for the student.

EXAMPLE 2. Find the positive integral solutions of the equation

$$13x - 9y = 151.$$

Divide by 9, the smaller coefficient ; then

$$x + \frac{4x}{9} - y = 16 + \frac{7}{9};$$

$$\therefore \frac{4x - 7}{9} = 16 - x + y$$

= an integer.

Multiply by 2 ; then

$$\frac{8x - 14}{9} = \text{an integer ;}$$

that is,

$$x - 1 - \frac{x + 5}{9} = \text{an integer ;}$$

$$\therefore \frac{x + 5}{9} = \text{an integer}$$

= p , suppose ;

$$\therefore x = 9p - 5.$$

By substituting in the original equation,

$$y = 13p - 24.$$

These two results furnish the *general solution* of the equation.

By giving to p any positive integral value greater than 1 we obtain positive integral values of x and y .

Thus we have

$$\left. \begin{array}{l} p = 2, 3, 4, 5, \dots \\ x = 13, 22, 31, 40, \dots \\ y = 2, 15, 28, 41, \dots \end{array} \right\}$$

the number of solutions being infinite.

As before, the solution may be illustrated graphically. It is obvious that the graph of $13x - 9y = 151$ has a positive intercept on the axis of x and a negative intercept on the axis of y . Hence the graph will lie in the first, fourth, and third quadrants. Since that portion which lies in the first quadrant can be produced to an infinite distance, it follows that there is no limit to the number of points which may have positive integral coordinates.

EXAMPLE 3. *In how many ways may £6 be paid in half-crowns and shillings, using both kinds of coins or only one?*

Let x be the number of half-crowns, y the number of shillings; then

$$2\frac{1}{2}x + y = 120,$$

or

$$2x + \frac{x}{2} + y = 120;$$

$$\therefore \frac{x}{2} = \text{some integer } p,$$

that is,

$$x = 2p. \dots\dots\dots(1)$$

Also

$$\begin{aligned} y &= 120 - \frac{5x}{2} \\ &= 120 - 5p. \dots\dots\dots(2) \end{aligned}$$

Since the sum may be paid in half-crowns alone, or in shillings alone, a zero value for x or y is not excluded. Hence from (1) and (2) we see that p may have the values 0, 1, 2, 3, ... 24.

Thus there are 25 ways of paying £6, using no coins except half-crowns and shillings.

EXAMPLE 4. *A man spent £4. 1s. in buying fowls at 2s. 6d., ducks at 3s. 6d., and pheasants at 4s. The number of birds bought was 25; how many were there of each?*

Suppose there were x fowls, y ducks, and z pheasants; then

$$2\frac{1}{2}x + 3\frac{1}{2}y + 4z = 81;$$

or

$$5x + 7y + 8z = 162. \dots\dots\dots(1)$$

Also

$$x + y + z = 25. \dots\dots\dots(2)$$

Eliminating x , we have

$$2y + 3z = 37.$$

This equation may be solved as before, but we shall here give a different method of solution.

The equation is obviously satisfied by $y = 8, z = 7$.

$$\therefore 2 \times 8 + 3 \times 7 = 37.$$

By subtraction,

$$2(y - 8) + 3(z - 7) = 0;$$

$$\therefore \frac{y - 8}{3} = \frac{7 - z}{2};$$

that is, $y - 8$ is the same multiple of 3 that $7 - z$ is of 2.

$\therefore$ we may put $y - 8 = 3p, 7 - z = 2p$, where p is an integer;

that is,

$$y = 3p + 8, \quad z = 7 - 2p.$$

Here

p may have the values 0, 1, 2, 3.

$$\therefore y = 8, 11, 14, 17;$$

$$z = 7, 5, 3, 1;$$

and from (2),

$$x = 10, 9, 8, 7.$$

EXAMPLES XLVII. c.

Solve in positive integers :

- | | | |
|---------------------|-----------------------|-----------------------|
| 1. $5x + 3y = 41.$ | 2. $7x + 4y = 85.$ | 3. $2x + 5y = 36.$ |
| 4. $7x + 11y = 58.$ | 5. $12x + 65y = 640.$ | 6. $11x + 13y = 390.$ |

Solve the following equations in positive integers, and verify the solutions graphically :

- | | | |
|--------------------|--------------------|---------------------|
| 7. $9x + 4y = 35.$ | 8. $3x + 5y = 56.$ | 9. $4x + 11y = 70.$ |
|--------------------|--------------------|---------------------|

Find the general solution in positive integers, and the least values of x and y which satisfy the equations :

- | | | |
|---------------------|----------------------|-----------------------|
| 10. $6x - 13y = 8.$ | 11. $5y - 7x = 29.$ | 12. $8x - 21y = 38.$ |
| 13. $8x - 7y = 31.$ | 14. $7x - 11y = 34.$ | 15. $10x - 13y = 46.$ |

16. A man spends £5. 10s. in buying two kinds of books at 3s. 6d. and 6s. each respectively : how many of each kind does he buy ?
17. In how many ways can £3. 2s. 6d. be paid in shillings and half-crowns, including zero solutions ?
18. Divide 152 into two parts so that one may be a multiple of 7 and the other of 12.
19. Find two fractions, having 7 and 11 for their denominators, such that their sum is $1\frac{34}{77}$.
20. What is the simplest way for a person who has only florins to pay 13s. 6d. to another who has only half-crowns ? In how many ways can the payment be made ?
21. A dealer in furniture has £183 to spend in buying tables and sofas costing £4. 12s. and £5 each respectively. How many of each can he buy ?
22. Divide 112 into two parts one of which when divided by 3 leaves remainder 2, and the other divided by 8 leaves remainder 7.

Solve the following pairs of simultaneous equations in positive integers :

- | | |
|-------------------------|----------------------------|
| 23. $2x + 5y + z = 21.$ | 24. $11x - 3(y + z) = 17,$ |
| $x - y + 2z = 11.$ | $4x - 6y + 3z = 25.$ |
25. A farmer buys 36 animals consisting of rams at £4, pigs at £2, and oxen at £17 : if he spends £214, how many of each does he buy ?
26. Given that $x = h$, $y = k$ is one solution of the equation $ax + by = c$, shew that the general solution is of the form

$$x = h + bp, \quad y = k - ap,$$

where p is an integer.

MISCELLANEOUS EXAMPLES X.

[The following Examples are arranged in three sets. I. may be taken after Chap. XLI; II. after Chap. XLIII.; III. after Chap. XLVII.]

I. (After Chap. XLI.)

1. Find the divisor when $(4a^2 + 7ab + 5b^2)^2$ is the dividend, $8(a + 2b)^2$ the quotient, and $b^2(9a + 11b)^2$ the remainder.

2. Resolve $4a^2(x^3 + 18ab^2) - (32a^5 + 9b^2x^3)$ into four factors.

3. Prove that $(y - z)^3 + (x - y)^3 + 3(x - y)(x - z)(y - z) = (x - z)^3$.

4. A man has a stable containing 10 stalls; in how many ways could he stable 5 horses?

5. Solve (i) $(x^2 - 5x + 2)^2 = x^2 - 5x + 22$; (ii) $x - 15\frac{3}{4} + \frac{5}{x - 15\frac{3}{4}} = 6$.

6. If α, β are the roots of $x^2 + px + q = 0$, shew that p, q are the roots of the equation $x^2 + (\alpha + \beta - \alpha\beta)x - \alpha\beta(\alpha + \beta) = 0$.

7. Simplify $\log \frac{133}{65} + 2 \log \frac{13}{7} - \log \frac{143}{90} + \log \frac{77}{171}$.

8. Find the coefficient of x^{14} in the expansion of $(2x^2 - 3x)^{10}$.

9. Find the factors of (i) $x^4 + 2x^2 + 9$; (ii) $9(a + b)^3 - 4(a + b)$.

10. If $x - \frac{1}{x} = y$, prove that $x^5 - \frac{1}{x^5} = 5y + 5y^3 + y^5$, and find a corresponding formula for $x^3 - \frac{1}{x^3}$.

11. Solve the equations:

$$(i) \frac{x+a}{x+b} = \frac{2x-a+b}{2x+a-b}; \quad (ii) x - cy = cx - y = c.$$

12. Write down the product of $(1+a)(1+b)$, and thence that of 1.01 and 1.02. If the last term is neglected, what is the resulting error per cent.?

13. If x is the harmonic mean between a and b , shew that it is also the harmonic mean between $x - a$ and $x - b$.

14. How many numbers greater than a million can be formed with the digits 2, 5, 0, 5, 1, 2, 5?

15. In an action between two battleships A and B , A fired 3 times as many shells as B . The total number of misses was 7 times the total number of hits. The number of B 's misses was 357, but B 's hits exceeded A 's hits by 66. What was the number of shells fired and the number of hits made by each?

16. Find the coefficient of x^r in the expansion of $\frac{3 - x + 2x^2}{(1 - x)^2}$.

17. Find A and B so that the equation

$$x^4 + x^3 + x^2 + x + 1 = (x^2 + Ax + 1)(x^2 + Bx + 1)$$

may be an identity.

18. If $\frac{y+z-x}{a} = \frac{z+x-y}{b} = \frac{x+y-z}{c}$, prove that

$$\frac{a+b+c}{x+y+z} = \frac{ay+bz+cx}{x^2+y^2+z^2},$$

19. Prove that $x^2 - 3x + 2$ is a common factor of

$$x^3 - 7x + 6, \quad 3x^3 - 7x^2 + 4, \quad \text{and} \quad x^4 - 3x^3 + 6x - 4.$$

For what values of x do these three expressions simultaneously vanish?

20. Draw the three graphs of

$$y=x, \quad y=3x, \quad \text{and} \quad y=(2x-5)(x+5)$$

between the values $x=-1$ and $x=+1$. Find by trial with the graphs, or otherwise, the two values of x at which the slopes of the third graph are equal to those of the other two respectively.

21. Find $\sqrt{14}$ to three decimal places by the Binomial Theorem, and check the result by logarithms.

22. Solve the simultaneous equations:

$$x+y+z=1, \quad ax+by+cz=0, \quad a^2x+b^2y+c^2z=0.$$

23. Prove by Mathematical Induction that

$$1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + \dots \text{ to } n \text{ terms} = \frac{1}{3}n(n+1)(n+5).$$

24. A man receives a pension starting with £100 the first year, but each year he receives 90% of what he received the previous year. Find, to the nearest penny, the total amount he receives in the first 6 years; find also the greatest amount he could possibly receive, even if he were to live for ever.

25. If $(a+b+c)x = (-a+b+c)y = (a-b+c)z = (a+b-c)w$, shew that

$$\frac{1}{y} + \frac{1}{z} + \frac{1}{w} = \frac{1}{x},$$

26. Find x and y from the equations $\frac{1}{2}(by-ax) = \frac{a^2x+b^2y}{b-a} = ab$.

27. If a, b are the roots of the equation $x^2 - 10x + 17 = 0$, calculate the values of

$$(i) (1-a)(1-b); \quad (ii) (1+a-a^2)(1+b-b^2).$$

28. In a certain town eggs are being sold at $2x$ pence a dozen, and in another town they are sold at x eggs for a shilling. By buying six dozen eggs in the latter and selling them in the former town a profit of 1s. is made; find the buying and selling prices of the six dozen eggs.

29. A number of squares are described whose sides are in G.P. Prove that the areas of the squares are also in G.P. The side of the $2m^{\text{th}}$ square is a feet and the side of the $2n^{\text{th}}$ square is b feet; find the area of the $(m+n)^{\text{th}}$ square.

30. At an election there are 4 candidates and 3 members to be elected, and an elector may vote for any number of candidates not greater than the number to be elected. In how many ways may an elector vote?

31. If $a+b+c=0$, prove that

$$\frac{a^2+b^2+c^2}{a^3+b^3+c^3} + \frac{2}{3} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = 0.$$

32. Given $\log 2 = \cdot 301030$, $\log 3 = \cdot 477121$, and $\log 7 = \cdot 845098$, find the logarithms of $0\cdot 005$, $6\cdot 3$, and $\left(\frac{49}{216} \right)^{\frac{1}{3}}$.

II. (After Chap. XLIII.)

33. The manufacturer's list price of an article exceeds the cost of making it by p per cent.; and it is sold to a retailer at a discount of q per cent. What is the manufacturer's percentage profit?

34. If $xx_1=b^2$, $y+y_1=2a$, and $xy_1=x_1y$, prove that

$$\frac{1}{x^2} = \frac{a}{b^2} \left(\frac{2}{y} - \frac{1}{a} \right).$$

35. If $x^2+3x+4=y$, find what value of y will give equal roots for x . Illustrate graphically.

36. If $t + \frac{1}{t} = x$, prove that

$$t^8 - \frac{1}{t^8} = \left(t - \frac{1}{t} \right) (x^7 - 6x^5 + 10x^3 - 4x).$$

37. A man rows down a river from a place A to a place B and back again from B to A without stopping in 2 hrs. 36 min. If the speed of the current is $1\frac{3}{4}$ miles per hour, and the distance from A to B is 3 miles, find the speed of the man in still water, and the times of his two journeys.

38. Find by logarithms the number of integral digits in $(7\cdot 2)^{16}$, and the number of digits in 3^{45} .

39. How many different arrangements, beginning with r and ending with n , can be made from the letters of the word *rotation*?

40. Find the general term of $\frac{1+5x}{1-2x-3x^2}$ when expanded in ascending powers of x .

41. If $\frac{x}{a+p} + \frac{y}{b+p} = 1 = \frac{x}{a+q} + \frac{y}{b+q}$, shew that $x = \frac{(a+p)(a+q)}{a-b}$, and find y .

42. If $a^3 + b^3 + c^3 = 3abc$, prove that either

$$a + b + c = 0, \quad \text{or} \quad a = b = c.$$

43. If α, β are the roots of the equation $x^2 + mx = 2 - m^2$, shew that

$$\alpha^3 - \beta^3 = 2(\alpha - \beta).$$

44. Find the square root of

$$(i) \ 27 - 7\sqrt{5}; \quad (ii) \ a^2 + x^2 + \sqrt{a^4 + a^2x^2 + x^4}.$$

45. Justify the following graphical construction for finding approximately 1.414 of any number up to 10. Join the origin to a point P whose coordinates are 10 and 14.14 (or 5 and 7.07), taking 1 inch as unit; then the ordinate of any point on OP is 1.414 times the corresponding abscissa. Read off from the diagram as correctly as possible to two places of decimals, 1.414×2 , 1.414×3.5 , 1.414×8.6 , $\frac{1}{1.414} \times 7.8$.

46. A and B, starting from the same place, make a journey of 56 miles; A starts 3 hours and 20 minutes after B, but travels 5 miles an hour faster. If they arrive at the same time, find the pace of each.

47. Shew that the number of men required to form a hollow square containing a men in the front rows and b men deep is $4b(a - b)$. Hence find the values of a and b for all the possible ways of arranging 1000 men in a hollow square.

48. Find correct to three decimal places the tenth root of the sum of the infinite series

$$1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \dots$$

49. Resolve $2(a^6 + b^6) - ab(a^2 + b^2)(2ab - 3a^2 + 3b^2)$ into five simple factors.

50. Divide $1 - x + x^2 - x^3 + \dots - x^{2p+1}$ by $1 + x^2 + x^4 + \dots + x^{2p}$.

51. The price of coffee being raised a pence per pound, b pounds fewer can be purchased for $2ab$ pence. How much per cent. is the increase of price?

52. Solve the equations:

$$(i) \ 4\sqrt{\frac{x}{x+2}} - 3\sqrt{1 + \frac{2}{x}} = 11;$$

$$(ii) \ \sqrt{5 - 2x} + \sqrt{15 + 3x} = \sqrt{26 - 5x}.$$

53. A man can row at the rate of a miles an hour in still water. He rows a distance of x miles down a river, which flows at the rate of b miles an hour, and back again. Find how long he will take, and shew that the time taken is longer than that which he would require to row $2x$ miles in still water.

54. If $2n+1$ quantities are in A.P., shew that the $(n-r+1)^{\text{th}}$, the $(n+1)^{\text{th}}$, and the $(n+r+1)^{\text{th}}$ terms are also in A.P.

55. Solve the following problem graphically :

X and Y are two towns 30 miles apart. A cyclist A leaves X at 2 p.m. and rides towards Y at the rate of $12\frac{1}{2}$ miles an hour ; a second cyclist B leaves Y at 2.5 p.m. and rides towards X at the rate of 13 miles an hour ; a third cyclist C leaves X at 2.15 p.m. and rides towards Y at the rate of 17 miles an hour. Shew that C will overtake A before A meets B, and find to the nearest half-mile how far B will be from Y when he meets C.

56. Shew that $\log_e \sqrt{\frac{x}{x-1}}$ may be expanded in the form

$$\frac{1}{2x-1} + \frac{1}{3(2x-1)^3} + \frac{1}{5(2x-1)^5} + \dots$$

57. If $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$, prove that $(x+y+z)^3 = 27xyz$.

58. If $x = (b-c)(a-d)$, $y = (c-a)(b-d)$, $z = (a-b)(c-d)$, find the value of $x^3 + y^3 + z^3 - 3xyz$.

59. If $x\sqrt{a^2 - y^2} + y\sqrt{a^2 - x^2} = a^2$, then $x^2 + y^2 = a^2$.

60. Divide 1 by $(1-x)^2$ so as to obtain a quotient in ascending powers of x . What is the remainder after n steps of the division have been performed ? Deduce the sum of $1 + 2x + 3x^2 + 4x^3 + \dots$ to n terms.

61. If x may have any real value, find the least value of $\frac{x^2 + 4}{x^2 + 4x + 4}$.

62. Find the coefficient of x^3 in the expansion of $(1-x)^n(1-x^2)^{2n}$.

63. A sells his motor car, which cost him £378, to B, who in turn sells it to C for £512. Given that A and B each make the same profit per cent. on their outlay, find to the nearest shilling the price for which A sells the car.

64. Draw a graph of $y = \frac{6x}{1+x^2}$ from $x=0$ to $x=9$, paying special attention to the shape of the curve between $x=0$ and $x=2$.

65. If a, b, c are any three numbers whose sum is zero, prove that the square of the sum of their products two at a time is equal to the sum of the squares of these products.

66. Express $(x+2)(x+3)(x+4)(x+5) - 15$ as the product of two quadratic factors.

67. Prove that $(x+y+z)^3 - (x^3 + y^3 + z^3) \equiv 3(y+z)(z+x)(x+y)$.

Thence, or otherwise, prove that

$$a(x^3 + y^3 + z^3) + b(x+y)(y+z)(z+x) + cxyz$$

is divisible by $x+y+z$ if $3a - b + c = 0$.

68. If $a : b = c : d$, prove that

$$(i) \left(\frac{1}{a} + \frac{1}{d} \right) - \left(\frac{1}{b} + \frac{1}{c} \right) = \frac{(a-b)(a-c)}{abc};$$

$$(ii) 4(a+b)(c+d) = bd \left(\frac{a+b}{b} + \frac{c+d}{d} \right)^2.$$

69. Solve the following pairs of equations :

$$(i) \begin{aligned} xy + x + y &= 11, \\ x^2y + xy^2 &= 30; \end{aligned}$$

$$(ii) \begin{aligned} x^4 - x^2y^2 + y^4 &= 117, \\ x^2 + xy\sqrt{3} + y^2 &= 39. \end{aligned}$$

70. Find the 13th term in the expansion of $(2^8 + 2^6x)^{\frac{11}{2}}$.

71. Express $\frac{5-x}{(1-x)(1+x^2)}$ in partial fractions.

72. Find the sum of the infinite series $x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$.

III. (After Chap. XLVII.)

73. To complete a piece of work A takes m times as long as B and C together; B takes n times as long as C and A together, and C takes p times as long as A and B together. Find the relation between m , n , and p .

74. Solve the equation $\frac{a}{x-b} + \frac{b}{x-a} = \frac{2p}{x-p}$ where $2p = a + b$.

75. The perimeter of a rectangular table is 200 inches; its area is halved when a strip 6 inches wide is cut off all round; find the dimensions of the original table to an accuracy of one-tenth of an inch.

76. Form the quadratic equation whose roots are the squares of those of $ax^2 + 2bx + c = 0$; and shew that the equation can be put in the form $(ax + c)^2 = 4b^2x$. Can you give a reason for this result?

77. Plot on as large a scale as you can the graph of $y = (1.5)^x$ for values of x between 0 and 6, using the Tables when necessary.

From the curve verify that $(1.5)^{\frac{3}{2}}(1.5)^{\frac{10}{3}} = (1.5)^{\frac{29}{6}}$.

78. Express $\frac{3x^2 - 11}{(x-2)^2}$ in partial fractions.

79. Find the value of $\frac{2\pi k(t_2 - t_1)}{\log r_2 - \log r_1}$, given that

$$\pi = 3.142, \quad k = 0.74, \quad t_1 = 69.4, \quad t_2 = 82.3, \quad r_1 = 1.25, \quad r_2 = 1.55.$$

80. If

$$\frac{p}{bc - a^2} = \frac{q}{ca - b^2} = \frac{r}{ab - c^2},$$

prove that

$$\frac{a}{qr - p^2} = \frac{b}{rp - q^2} = \frac{c}{pq - r^2}.$$

81. Express $\sqrt{2x^2 + xy - 6y^2} \cdot \sqrt{3x^2 + 5xy - 2y^2} \cdot \sqrt{6x^2 - 11xy + 3y^2}$ in its simplest form.

82. Simplify $\frac{(p^2 + q^2)(x + y)^2 + 2(px - qy)(qx - py)}{(p^2 - q^2)(x^2 + y^2)}$.

83. If $9x^4 - 12x^3y + Px^2y^2 + 4xy^3 + y^4$ is a perfect square, find P.

84. Find x and y from the equations :

$$x^2 - 2xy + y^2 + 2x + 2y - 3 = 0 = y(x - y + 1) + x(x - y - 1).$$

85. A, B, C, D are four stations on a railway, the distances AB, BC, CD being 10 miles, 10 miles, and 8 miles respectively. The following is an extract from a time-table :

<i>Up Train.</i>	<i>Down Train.</i>
A, dep., 7.57 a.m.	D, dep., 8.28 a.m.
B, dep., 8.18 a.m.	C, —
C, dep., 8.40 a.m.	B, —
D, arr., 8.55 a.m.	A, arr., 9.10 a.m.

Draw graphs to shew the positions of the trains at any intermediate time, assuming that each runs at a uniform speed between the stations, and that the up train stops 3 minutes at each of the stations B, C. When and where do the trains pass each other? Shew that the down train passes B just as the up train reaches D.

86. A man borrows £20 from a money lender and he has to repay £24 in monthly instalments of £2, the first to be paid at the end of the first month. Reckoning simple interest at the rate of r per cent. per annum, find the sum to which £20 amounts in a year, and shew that the sums repaid, together with interest on repayments, amount to $\pounds \left(24 + \frac{11r}{100}\right)$. He imagines that he is paying 20% interest; determine the actual rate.

87. Shew that 1030301 is a complete cube in any scale whose radix is greater than 3.

88. If $y^2 + yz + z^2 = a$, $z^2 + zx + x^2 = b$, $x^2 + xy + y^2 = c$,

shew that $3x = t + \frac{b + c - 2a}{t}$, $a + b + c = t^2 + \frac{p}{t^2}$,

where $t = x + y + z$, and $p = a^2 + b^2 + c^2 - bc - ca - ab$.

Solve the equations for x, y, z when $a = 3, b = 13, c = 7$.

89. If $x + \frac{1}{y} = 1$, and $y + \frac{1}{z} = 1$, shew that $z + \frac{1}{x} = 1$.

90. If $a + b + c = 0$, prove that

$$\frac{a^4}{b^3 + c^3 - 3abc} + \frac{b^4}{c^3 + a^3 - 3abc} + \frac{c^4}{a^3 + b^3 - 3abc} = 0.$$

91. Find what values, if any, of x and y satisfy all three of the equations :

$$2x + 3y = 5, \quad y = 3x + 1, \quad \frac{x}{2} + \frac{y}{8} = 1.$$

Illustrate by drawing graphs of the equations.

92. Simplify the following expressions :

(i) $2(2 + \sqrt{3})(\sqrt{6} - \sqrt{2})\sqrt{2 - \sqrt{3}}$;

(ii) $x^3 + y^3 + z^3 + 3(x + y + z)(yz + zx + xy) - (x + y + z)^3$.

93. Solve the equation

$$\frac{2x - 11}{x - 2} + \frac{x + 4}{x - 3} = \frac{x - 5}{x + 2} + \frac{2x + 9}{x + 1}.$$

94. A party of four people is to be chosen from nine, among whom are A and his wife and B and his wife. A , if invited at all, must be invited with his wife ; similarly B and his wife must be invited together, if at all. In how many ways can the party be chosen ?

95. If a farthing is put out at compound interest for 1000 years at 5%, how many digits will be required to express the amount in pounds ?

96. In a certain machine P kilograms is the effort required to move a load of W kilograms. The following values were obtained experimentally :

$$\begin{array}{ccccccc} P = & 10.6, & 12.2, & 15.2, & 18.4, & 21.6, & 24.6, & 27.8 ; \\ W = & 5, & 10, & 20, & 30, & 40, & 50, & 60. \end{array}$$

Plot these values, and assuming the relation between P and W to be of the form $P = aW + b$, find the values of a and b .

97. If $x = a(b - c)$, $y = b(c - a)$, $z = c(a - b)$, prove that

$$\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 + \left(\frac{z}{c}\right)^3 = \frac{3xyz}{abc}.$$

98. A man receives $\frac{x}{y}$ of 10s. and afterwards $\frac{y}{x}$ of 10s. He then gives away a sovereign ; shew that he cannot lose by the transaction.

99. Solve the equation

$$\sqrt{x + a - c} + \sqrt{x + b - c} = \sqrt{c - a} + \sqrt{c - b}.$$

100. Eliminate x and y from the equations :

$$x + y = a, \quad x^2 + y^2 = b^2, \quad x^3 + y^3 = c^3.$$

101. An express leaving P at 3 p.m. reaches Q at 6 p.m. ; a slow train leaving Q at 1.30 p.m. arrives at P at 6 p.m. ; if both trains are supposed to travel at a uniform speed, find graphically the time when they will meet. Shew also that the time does not depend upon the distance between P and Q .

102. Prove that the sum of n terms of the series

$$1, \quad 1+r, \quad 1+r+r^2, \quad 1+r+r^2+r^3, \dots$$

is

$$\frac{n - (n+1)r + r^{n+1}}{(1-r)^2}.$$

103. On a bookstall there are 2 copies of one work, 3 of another, and 4 of a third; in how many ways can a purchaser make a selection by taking one or more from the 9 volumes?

104. The value of P has to be found from the formula $P = k \frac{l}{t^2 r^3}$, where k is a constant, and l, t, r are found by experiment. If there is an error of 0.4% too much in the value of l , 1.5% too little in the value of t , and 0.2% too much in the value of r , find the percentage error in the value of P .

105. Using Detached Coefficients, find the first four terms of

$$(1 + 2x - 4x^2 + x^3)(2 - x^2 + 3x^3 + x^5).$$

106. If $xy^{p-1} = a$, $xy^{q-1} = b$, $xy^{r-1} = c$, prove that

$$(q-r) \log a + (r-p) \log b + (p-q) \log c = 0.$$

107. If $(a+b+c+d)(bc+ca+ab) = abc + abd + acd + bcd$, shew that

$$(b+c)(c+a)(a+b) = 0.$$

108. Find the square root of 223141 in the scale of five.

109. Rationalize the equation

$$(y+z-x)^{\frac{1}{3}} + (z+x-y)^{\frac{1}{3}} + (x+y-z)^{\frac{1}{3}} = 0.$$

From the resulting equation, shew that

$$(x+y+z)^4 - 27(x^2+y^2+z^2)^2 + 54(x^4+y^4+z^4) = 0.$$

110. In how many ways can 5 men take their places in an empty railway carriage with 8 seats, if one of them must always have a corner seat, and another must travel facing the engine?

111. If P and Q vary respectively as $y^{\frac{1}{2}}$ and $y^{\frac{1}{3}}$ when z is constant, and as $z^{\frac{1}{2}}$ and $z^{\frac{1}{3}}$ when y is constant, and if $x = P + Q$, find the equation between x, y, z ; it being known that when $y = z = 64$, $x = 12$; and that when $y = 4z = 16$, $x = 2$.

112. The sum of n terms of an A.P. is s ; shew that the sum of their squares is

$$\frac{s^2}{n} + \frac{1}{12} n(n^2 - 1)d^2,$$

where d is the common difference.

113. Given that $x^4 + 4x^3 + px^2 + qx + 9$ is the square of $x^2 + ax + b$, find all the possible values of a , b , p , and q .

114. If $a + b + c = 0$, prove that $\sum a \left(\frac{b^3 - c^3}{b - c} \right) = 0$.

115. Sum to n terms the series whose n^{th} term is $2^n + n(n - 1)$.

116. If nC_r is the number of combinations of n things r at a time, prove by general reasoning that

$${}^{n+2}C_{r+1} = {}^nC_{r+1} + {}^nC_{r-1} + 2 \times {}^nC_r.$$

117. Solve the equations :

$$(i) \frac{1}{x+a} - \frac{1}{x+b} - \frac{1}{x+a+b} + \frac{1}{x+2b} = 0;$$

$$(ii) \frac{x-2a}{b+c-a} + \frac{x-2b}{c+a-b} + \frac{x}{a+b+c} = 3.$$

118. Shew that the sum of all the products in a multiplication table going up to n times n is $\frac{n^2(n+1)^2}{4}$.

119. Shew that the coefficient of x^{2n+1} in the expansion of $(1 + x + x^2 + \dots + x^{n-1})^3$ is $\frac{(n-2)(n-3)}{2}$. Verify this result when $n=3$.

120. To provide for his two infant sons, a man left by his will two sums of money as separate investments at different rates of interest, on the condition that the principal sums with simple interest were to be paid over to his sons when the amounts were the same. After 5 years the first sum amounted to £451, and after 15 years to £533. After 10 years the second sum amounted to £432, and after 20 years to £544. Draw graphs from which the amounts may be read off for any year, and find after how many years the sons were entitled to receive their legacies.

Also determine from the graphs what the original sums were at the father's death.

EXAMPLES FROM RECENT EXAMINATION PAPERS.

A. ELEMENTARY

1. Write down the factors of

$$9x^2 - 25y^2; \quad x^2 - 2x - 63; \quad x^2 - y^2 + z^2 - 2xz; \\ a^2 + 2bc - ab - 2ac. \quad (\text{Oxf. Sch. Certif.})$$

2. (i) Solve the equation

$$\frac{1}{3}(x+1) - \frac{1}{2}(3x - \frac{1}{3}) + (x + \frac{1}{2}) = 0.$$

(ii) From the equation $1+a=(1+b)\frac{1+pt}{1+qt}$, find t in terms of the other quantities. (Camb. Sch. Certif.)

3. If $x=4a^2+3ab+b^2$ and $y=2(a^2-2ab-b^2)$, express x^2-y^2 as the product of four factors in terms of a and b . (Oxf. Sch. Certif.)

4. (a) If $W=C^2R+\frac{t^2}{R}$, find, by logarithms, the value of W when $R=0.044$, $C=41.1$ and $t=1.8$.

(b) Show, without the use of tables, that

$$2(\log \sqrt{125} + \log 27 - \log \sqrt{1000}) = 3(\log 9 - \log 2). \quad (\text{Joint Matric. Bd.})$$

5. If the same temperature is represented either by x° Fahrenheit or y° Centigrade, x and y are connected by the relation $ax-by=160$ where a and b are constants. Knowing that 32° Fahrenheit is the same temperature as 0° Centigrade, and that 212° Fahrenheit is the same as 100° Centigrade, find the values of a and b . (Oxf. Sch. Certif.)

6. A man rented a certain number of acres for £72. He worked 2 acres himself, and, by letting the rest for 8s. per acre more than he paid for it, he received for this portion an amount equal to the rent of the whole. How many acres did he rent altogether? (Oxf. and Camb. Sch. Certif.)

7. The first term of a Geometrical Progression is 2, and the second is 2.4. Find the sum of n terms of the progression, proving any formula that you may use.

Find also the least value of n for which this sum is greater than 2,000. (Lond. Matric.)

8. (i) Factorize :

$$(a) x^2 - 4x - 12; \quad (b) x^3 - 7x - 6.$$

(ii) Simplify $\frac{4}{x-4} - \frac{2}{x+5} + \frac{6(x-2)}{x^2+x-20}$. (Cent. Welsh Bd.)

9. Express each of the following as a single fraction in its lowest terms :

$$(a) x - 2 + \frac{2x+1}{x}, \quad (b) \frac{x^2+1}{x^2-1} + \frac{x-1}{x+1}, \quad (c) \frac{2}{x-2} - \frac{x}{(x-1)(x-2)}. \quad (\text{Joint Matric. Bd.})$$

10. (i) Given $\log 5 = 0.6990$, find $\log 2$ without the use of Tables. Given, in addition, $\log 7 = 0.8451$, find the value of $\log 280$. Show your working.

(ii) Calculate the value of $\sqrt{\frac{x^2 - 4y^2}{xy}}$ if $x = 48.73$, $y = 17.16$.

(Lond. Matric.)

11. If $c = \frac{ax - 2by}{ax + 2by}$,

(i) express x in terms of a , b , c and y ,

(ii) express b in terms of a , c , x and y .

(Oxf. and Camb. Sch. Certif.)

12. Solve the equations :

$$(i) \quad 3 \left(\frac{1}{x} - \frac{1}{4} \right) - 4\frac{1}{2} \left(\frac{1}{3} - \frac{1}{x} \right) = 5 \left(\frac{1}{x} - \frac{1}{5} \right);$$

$$(ii) \quad \frac{3x + 2y}{2x + 3y} = \frac{59}{71}, \quad 3x - y = 2. \quad (\text{Oxf. Sch. Certif.})$$

13. Draw the graph of $y = \frac{7}{2} - x - x^2$ for values of x from -3 to $+2$.

Use your graph to find :

(a) the range of values of x , for which x and y are both positive ;

(b) the roots of the equation $4x^2 + 6x - 1 = 0$. (Joint Matric. Bd.)

14. A car made a run of 195 miles in a certain time. If its average speed had been 4 miles per hour greater it would have taken an hour less for the journey. How long did it take ? (Scot. Cert.)

15. (a) Solve the equations :

$$\left. \begin{aligned} \frac{2x + y}{x + 3y} &= -\frac{4}{3} \\ 7x + 36y &= -8\frac{1}{2} \end{aligned} \right\};$$

(b) If $\frac{x}{y} = \frac{4}{3}$, find the value of $\frac{3x - 2y}{3x + 2y}$.

(Oxf. and Camb. Sch. Certif.)

16. (a) Prove, by complete factorization or otherwise, that

$$\begin{aligned} (x^2 - 7x + 12)(x^2 + 7x + 12) &= (x^2 - x - 12)(x^2 + x - 12) \\ &= (x^2 - 16)(x^2 - 9). \end{aligned}$$

(b) Prove that $2x + 3a$ is a factor of $4x^3 + 4ax^2 - 15a^2x - 18a^3$ and find the other factors. (Scot. Cert.)

17. Simplify :

$$(i) \quad \left\{ \frac{x}{a^2} + \frac{a}{x^2} \right\} \div \left\{ \frac{x}{a} - 1 + \frac{a}{x} \right\};$$

$$(ii) \quad \frac{2x - 1}{x^2 - x - 6} - \frac{x + 2}{2x^2 - 7x + 3} + \frac{x - 3}{2x^2 + 3x - 2}.$$

(Camb. Sch. Certif.)

18. If $\frac{x^2}{y^2} = \frac{27.43}{6.97}$, use logarithms to find $\frac{x}{y}$ and $\frac{y}{x}$.

Also find to three places of decimals $(0.9)^7$. (Oxf. Sch. Certif.)

19. If the sum of two positive whole numbers is equal to the difference of their squares, show that the numbers are consecutive.

The sum of the squares of two numbers is less by 1 than twice the difference of their squares and the sum of the numbers is less by 1 than four times their difference. Find the numbers. [There are two solutions.]

(Lond. Matric.)

20. (i) Find the sum of 27 terms of the Arithmetical Progression whose third term is 27 and whose fourteenth term is 9.

(ii) The sum of three consecutive terms of a Geometrical Progression is 52 and the square of the middle term is four times the last term. Find the values of these terms. (Camb. Sch. Certif.)

21. Draw the graph of $y = \frac{2(14 - x^2)}{x + 11}$ for values of x from $x = -2$ to $x = +4$, using one inch for unit on both axes.

Using the same axes and scale, draw the graph of $2x + 3y = 8$.

Prove that the values of x at the points where these graphs meet ought to satisfy the equation $2x^2 - 7x + 2 = 0$, and read off their values approximately from your diagram. (Camb. Sch. Certif.)

22. What is the result of dividing the product of $x^2 + x - 12$ and $x^2 - 3x - 4$ by $x^2 - 16$? (Lond. Matric.)

23. Solve the equation :

$$\frac{2}{3(3x-2)} - \frac{3}{2(3x+2)} = \frac{3}{9x^2-4}.$$

Verify the correctness of your answer by substituting in the original equation. (Oxf. and Camb. Sch. Certif.)

24. (i) If $\log_{10} y - \log_{10} \sqrt{x} = 1$, find the value of y when $x = 9$, without using tables.

(ii) With the help of logarithmic tables, evaluate

$$\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

when $a = 351$, $b = 41$, $c = 320$, and $2s = a + b + c$.

(Oxf. and Camb. Sch. Certif.)

25. The cost of making a spherical ball varies as the cube of its radius, and the cost of painting the ball varies as the square of its radius. If a painted ball of radius 3 ft. costs £5. 1s. 3d., and one of radius 1 ft. 6 in. costs 13s. 6d., find the cost of one whose radius is 2 ft. (Camb. Sch. Certif.)

26. Find the value, in terms of a and b , of $\sqrt{\frac{z-3a}{z-2b}}$ when $z = \frac{6ab}{3a+2b}$. (Oxf. Sch. Certif.)

27. An article is sold at a profit of 18 per cent. on the cost price. If both the cost and selling prices were raised by £5 the profit would be 16 per cent. Find the cost price. (Oxf. Sch. Certif.)

28. (a) Solve the equation :

$$10(x+1) + \frac{10}{x+1} = 29.$$

(b) Find the positive integral value of x for which the value of $2x^2 - 9x$ is nearest to 400. (Joint Matric. Bd.)

29. Factorize :

(a) $x^2 + 13x - 48$; (b) $8x^2 + 11x - 10$; (c) $2x^3 + 16y^3$.

(Joint Matric. Bd.)

30. (i) Simplify $\frac{a^2 + b^2}{(a+b)^2} + \frac{\frac{2}{ab}}{\left(\frac{1}{a} + \frac{1}{b}\right)^2}$.

(ii) Solve the equations :

(a) $\frac{x+y}{x-y} = \frac{5}{3}$; $x + 5y = 36$, (b) $\frac{1}{x} - \frac{1}{7} - \frac{1}{8} = \frac{1}{x-15}$. (Scot. Cert.)

31. (i) Simplify $\left(\frac{y^2}{z}\right)^3 \left(\frac{z}{x^3}\right)^2 \left(\frac{x^3}{y^2}\right)$,

and evaluate when $x = -1$, $y = -2$, $z = 4$.

(ii) Evaluate by logarithms

$$M \frac{r-1}{r+1},$$

where $M = 64.17$, $r = 1.544$.

(Cent. Welsh Bd.)

32. If $xy + 5x - 4y - 16 = 0$, express (i) x in terms of y ; (ii) y in terms of x .

If in addition it is known that $x = -3y$, find the numerical values of x and y . (Lond. Matric.)

33. A boy was engaged to work for 30 days on the understanding that he received 5s. for each day he worked and was fined 6d. for each day he was idle. At the end of the time his net receipts were £5. 11s. 6d. How many days did he work ? (Oxf. and Camb. Sch. Certif.)

34. Plot the graph of $y = x - \frac{2}{x}$ for values of x between $\frac{1}{2}$ and 3 and between $x = -\frac{1}{2}$ and -3 , using the same scale for x and y . The scale chosen must be large enough for the graph to cover at least half the height of the graph paper.

Find from the graph the values of x for which $y = 1.75$. (Lond. Matric.)

35. (i) Sum the series $\frac{3}{8} + 1\frac{5}{16} + 2\frac{1}{4} + \dots + 21\frac{15}{16}$.

(ii) The third term of a G.P. exceeds the second by $\frac{9}{14}$, and the second exceeds the first by $\frac{3}{7}$. Find the common ratio and the first term.

(Cent. Welsh Bd.)

36. Resolve into their simplest factors :

(a) $4x^2 - 33xy - 27y^2$; (b) $(y^2 + 1)^2 - 4y^2$; (c) $abc - abd + ac^2 - acd$.

(Joint Matric. Bd.)

37. Solve the equations :

$$(i) \frac{x^2 - x + 1}{x - 1} - \frac{x^2 + x + 1}{x + 1} = \frac{14x}{x^2 - 1};$$

$$(ii) \frac{x}{7} - \frac{y}{3} = \frac{2x}{3} - 2y - 1 = \frac{y}{2} - 2. \quad (\text{Oxf. Sch. Certif.})$$

38. Use logarithms to calculate the cube roots of 3.875 and of 0.03875. Find also the cube roots of 3875, 38750, and 0.00003875. (Lond. Matric.)

39. A man set apart £18 for a certain length of holiday, but, wishing to extend it without extra expense, found that he could manage an additional 4 days by reducing his expenditure by 3s. per day throughout. What length of holiday did he originally plan? (Oxf. and Camb. Sch. Certif.)

40. If $\frac{a}{b} = \frac{x}{y}$, prove that each of these fractions is equal to $\frac{a+x}{b+y}$.

Prove also that

$$\frac{3a^2 + 5ab + 2b^2}{3a^2 - 5ab + 2b^2} = \frac{3x^2 + 5xy + 2y^2}{3x^2 - 5xy + 2y^2}. \quad (\text{Scot. Cert.})$$

41. (a) Make the following expressions into complete squares by adding a single term to each :

$$(i) p^2 - \frac{3pq}{2}. \quad (ii) 4x^2 + 6x.$$

(b) One factor of the expression

$$x(x-1)(x-2) - (x-3)(x+20)$$

is $x-4$. Find the remaining quadratic factor. (Cent. Welsh Bd.)

42. The third term of a Geometrical Progression is positive, the sum of the first, third and fifth terms is 91, and the product of the second and fourth terms is 81. Find the values of these five terms.

(Camb. Sch. Certif.)

43. If $x = a - 1$, $y = 2$, $z = -(a + 1)$, find the value of $x^3 + y^3 + z^3$ in terms of a . (Oxf. Sch. Certif.)

44. Factorize : (i) $3a^3b - 12ab^3$; (ii) $x^2 + xy + y - 1$.

(Oxf. and Camb. Sch. Certif.)

45. Solve the equations :

$$(a) \left. \begin{array}{l} x + y = 6 \\ 3y - 2x = 3 \end{array} \right\}; \quad (b) \frac{1}{x} + \frac{a}{b} = \frac{b}{a}; \quad (c) 2x^2 + 5x = 3.$$

(Joint Matric. Bd.)

46. Explain clearly what is meant by the logarithm of a number; and show that in the common system of logarithms the logarithm of 315 is a number between +2 and +3, and the logarithm of 0.0315 a number between -1 and -2.

Find the cube root of 0.0315; and the value (correct to the nearest shilling) of £25.3 (1.03)⁶. (Scot. Leaving.)

47. (a) Express as an algebraical equation the statement: " z varies inversely as y and directly as the square of x ."

(b) If x miles per hour represents the average speed of a train for a journey of 200 miles, express in words the following equation:

$$\frac{200}{x} - 1 = \frac{200}{x + 10}. \quad (\text{Joint Matric. Bd.})$$

48. Draw the graph of

$$y = (x + 1)(x + 2)$$

for values of x from -3 to $+1.5$.

Between what values of x within this range does the expression $(x + 1)(x + 2)$ decrease in value as x increases? Use your graph to find for what values of x the expression $x^2 + 3x + 2$ is equal to 1.6 .

Draw (and label) the line whose points of intersection with the curve would give solutions of the equation $x^2 + 3x + \frac{3}{2} = 0$. (Cent. Welsh Bd.)

49. The weight of a metal rod varies jointly as its length and the square of its diameter. When the length is 8 inches and the diameter $1\frac{1}{2}$ inches the weight is 12 oz. Find the length of a rod of the same metal of diameter $2\frac{1}{4}$ inches, whose weight is 27 oz. (Camb. Sch. Certif.)

50. Resolve into their simplest factors:

(a) $2x^2 - xy - y^2$; (b) $ax^2 - 2bxy - 2bx^2 + axy$; (c) $4p^2 - 4pq + q^2 - 4x^2$.
(Cent. Welsh Bd.)

51. (a) Correct the mistakes occurring on the right-hand sides of the following statements:

(i) $3^2 \times 3^3 = 9^5$; (ii) $2^9 \div 2^3 = 2^3$; (iii) $(2ab^3)^2 = 2a^2b^6$.

(b) Find the logarithms of 816.7 and 0.007319 and the number whose logarithm is 1.5791. (Joint Matric. Bd.)

52. Solve the equations:

(a) $\frac{20}{x} - 4 = \frac{104 - 10x}{4x^2}$;
(b) $\left. \begin{aligned} \frac{2}{3}(3x - 5) + \frac{3}{4}(4x - 5) &= 3x - 3\frac{3}{4} \\ \frac{3}{4}(x - 2y + 4) + \frac{4}{5}(2x - y - 5) &= 1 \end{aligned} \right\}$. (Scot. Cert.)

53. If x^2 and y^2 are not equal and not zero, and if

$$\frac{x^2 + y^2}{x^2 - y^2} - \frac{x^2 - y^2}{x^2 + y^2} = \frac{x + y}{x - y} - \frac{x - y}{x + y},$$

prove that $x^2 + y^2 = xy$. (Lond. Matric.)

54. Two rectangular tanks, A and B , standing on the same horizontal plane, are connected at their lower ends by a small pipe and the area of the base of A is three times that of B . The pipe is blocked up and the depth of water in A is 5 inches less than in B . When the obstruction is removed water flows from B to A until its depth in each tank is $15\frac{1}{4}$ inches. Find the original depth of the water in each tank. (Oxf. Sch. Certif.)

55. Find the first three terms and the last three terms in the quotient of $x^{11} - 11x + 10$ when divided by $x^2 - 2x + 1$. (Oxf. and Camb. Sch. Certif.)

56. Draw the graph of $y = \frac{4(9 - x^2)}{9 - x}$ for values of x from -3 to $+3$, using one inch for unit on both axes.

Explain how the roots of the equation $4x^2 - 3x - 9 = 0$ can be found from the graph, and read off approximate values for them. (Camb. Sch. Certif.)

57. Show that the equation

$$(x + 2y)(2x - y) + (x - y)(3x + 4y) = 22$$

is true when $x = 2$ and $y = 1$. Find any other values of x which make the equation true when $y = 1$ and any other values of y for which it is true when $x = 2$. (Lond. Matric.)

58. (a) Given that one of the factors of

$$10x^3 - x^2 - 33x + 18 \text{ is } 2x - 3,$$

factorize this expression fully.

(b) Express as a fraction in its lowest terms :

$$\frac{a^2 - 4x^2}{4a^2 - x^2} \times \frac{x(2a - x) + a(2a + x) - (4a^2 - x^2)}{x(a - 2x) + a(a + 2x) - (a^2 - 4x^2)} \cdot \frac{(3ax - 2a^2)(a + 2x)}{(2x^2 - 3ax)(2a + x)}.$$

(Scot. Cert.)

59. (a) A man's gross income is £ b . Find in pounds his net income after payment of tax at the rate of c shillings in the pound.

(b) Given that
$$pv = cd \left(1 + \frac{t}{273} \right),$$

express t in terms of the other quantities in the formula.

(Cent. Welsh Bd.)

60. Solve the equations :

(i) $3(x - 9)^2 - 2(x - 9) - 16 = 0;$

(ii)
$$\left. \begin{aligned} 3x + 2y &= 4xy \\ 2x + 3y &= 7 \end{aligned} \right\}.$$
 (Camb. Sch. Certif.)

61. (a) Prove that $\log_{10} \frac{a}{b} = \log_{10} a - \log_{10} b$.

(b) Given that $\log_{10} 2 = p$ and $\log_{10} 3 = q$, find in terms of p and q and without the use of tables the values of $\log_{10} 6$, $\log_{10} 5$, and $\log_{10} 24$.

(c) Find the value of V from the formula

$$V = 0.56 \sqrt{2g(H - h)},$$

where $g = 32.2$, $H = 125.3$ and $h = 12$.

(Cent. Welsh Bd.)

62. Draw, on the same diagram, the graphs of

$$y = x^3 \quad \text{and} \quad y = 2x^2 + 3x$$

from $x = -2$ to $x = 3$, and write down the coordinates of the points in which the curves intersect. What are the roots of the equation $x^3 - 2x^2 - 3x = 0$?
(Oxf. and Camb. Sch. Certif.)

63. A number consists of three digits whose sum is 12, and the unit's digit exceeds the hundred's digit by 2. If the digits are reversed and the number so formed is added to the original number the sum is 1212. Find the number.
(Oxf. Sch. Certif.)

EXAMPLES FROM RECENT EXAMINATION PAPERS

B. ADDITIONAL MATHEMATICS

1. Solve the equations :

$$(i) \quad 3x^2 - y^2 = 5x + 3y, \quad 3x + 2y = 1;$$

$$(ii) \quad \left. \begin{aligned} x - y + z &= 0 \\ 2x + y - 3z &= 0 \\ x^2 + 2y^2 + 3z^2 &= 9 \end{aligned} \right\}.$$

(Camb. Sch. Certif.)

2. (a) Assuming that $a^m \times a^n = a^{m+n}$ for all values of m and n , find meanings for $a^{\frac{2}{3}}$ and $a^{-\frac{2}{3}}$.

(b) Multiply $\sqrt{a^3} + 1 + \frac{1}{\sqrt{a^3}}$ by $\sqrt{a^3} - 1 + \frac{1}{\sqrt{a^3}}$.

(c) Find the square root of $a^{\frac{4}{3}} - 6a^{\frac{2}{3}} + 4a^{\frac{1}{6}} + 9 - 12a^{-\frac{1}{2}} + 4a^{-1}$.

(Joint Matric. Bd.)

3. If $\log_e x = A \log_e y$, prove that $\log_{10} x = A \log_{10} y$.

If r_1 is the reading of a barometer at sea level, the height h above sea level at which the reading is r_2 is given by

$$h = A \log_e \left(\frac{r_1}{r_2} \right),$$

where A is a constant. If the barometer reads 30 in. at sea level and 28 in. at a height of 1640 ft., find its reading at 3000 ft. (Lond. Matric.)

4. (a) When are quantities p, q, r, s said to be in continued proportion? If $p = 1.2$ and $s = 0.15$, find q and r .

(b) Show that if p, q, r, s are in continued proportion,

$$(q - r)^2 + (r - p)^2 + (s - q)^2 = (p - s)^2.$$

(Scot. Cert.)

5. Find the sum of 10 terms of the series

$$1 + 4 + 7 + 10 + 13 + \dots,$$

and show that the sum of n terms is equal to

$$\frac{3}{2} \left\{ \left(n - \frac{1}{6} \right)^2 - \frac{1}{36} \right\}.$$

For what values of n does the sum of n terms lie between 300 and 400? (Oxf. and Camb. Sch. Certif.)

6. Solve $\frac{2x^2 - 6x + 9}{(x - 2)^2} = y$ as a quadratic in x . From your result find the smallest possible value of y if x is real. (Oxf. Sch. Certif.)

7. Write down the first 5 terms of the expansion of $(1 - 3x)^{10}$ in ascending powers of x .

Hence find the value of $(0.997)^{10}$ to seven places of decimals.

(Camb. Sch. Certif.)

8. (a) Solve the equations :

$$2x^2 + 3y^2 = 11, \quad x^2 - xy + y^2 = 7.$$

(b) Prove that $x = \frac{c(a - b)}{a(b - c)}$ is a root of the equation

$$a(b - c)x^2 + b(c - a)x + c(a - b) = 0;$$

and find the other root.

(Scot. Cert.)

9. Without using tables, find the values of

- (i) $32^{-\frac{2}{5}}$, (ii) $\log(\sqrt[3]{10} \div \sqrt{10})$,
 (iii) $\log \frac{270}{64}$: you may assume here that $\log 2 = \cdot 3010$, $\log 3 = \cdot 4771$.
 (Oxf. and Camb. Sch. Certif.)

10.
$$\frac{3x+2}{(x-1)^2(2x-3)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{2x-3}.$$

Find A , B , and C .

(Oxf. Sch. Certif.)

11. (a) If a and b are the first and last terms of an arithmetical progression of $r+2$ terms, find the second and the $(r+1)$ th terms.

(b) Find the sum of n terms of the series $9+7+5+3+\dots$ and check the result by taking $n=3$.
 (Joint Matric. Bd.)

12. Find the values of the coefficient a and b if the expression

$$x^5 + x^4 + ax^3 + bx^2 + 2x + 8$$

is exactly divisible by $x^2 - x + 2$.

Give the quotient when a and b have these values.

(Scot. Cert.)

13. A merchant sent by post sample packets of two substances; the cost of the postage for two packets, one of each kind, was 1s. 11d. For £5 he could send 70 more packets of one kind than of the other. Find the postage on each packet.
 (Oxf. Sch. Certif.)

14. State and prove the Binomial Theorem for a positive integral value of the index.

If the expansion of $(1+ax)^8(1+bx)^9$ begins with the terms $1-x-8x^2$, find the values of a and b .
 (Lond. Matric.)

15. (a) Assuming that $a^m \times a^n = a^{m+n}$ for all values of m and n , prove that $(a^m)^n = a^{mn}$.

(b) Simplify
$$\frac{4^{-n+2} \times 2^{2n+1} \times 8^{n+1}}{2^{3n+2}}.$$

(c) Divide $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$ by $x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.

(Joint Matric. Bd.)

16. Solve the equations :

$$2x - y + 1 = 0, \quad \frac{1}{x} + \frac{3}{y} = 2.$$

(Oxf. and Camb. Sch. Certif.)

17. (a) Simplify the expression :

$$\log_{10} \left(\frac{5}{4} \right) + \log_{10} 14 - \log_{10} \left(\frac{7a}{3} \right),$$

and, without using the tables, find the value of a so that the expression is equal to -1 .

(b) Solve the equations :

$$\left. \begin{aligned} 2^{2x} 3^{-y} &= 8 \\ x - 2y &= 2 \end{aligned} \right\},$$

(obtaining the results to three significant figures.

(Cent. Welsh Bd.)

18. If $a : b = c : d$, prove that

$$(a - b) : (a + b) = (c - d) : (c + d).$$

Solve the equation : $\frac{x^2 - 3x + 4}{x^2 + 3x + 4} = \frac{x^3 - 3x^2 - 2x + 5}{x^3 + 3x^2 + 2x + 5}$. (Oxf. Sch. Certif.)

19. (i) If α, β are the roots of $ax^2 + bx - a = 0$, prove that

$$(a\alpha + b)(a\beta + b) = -a^2,$$

and find the equation whose roots are $a\alpha + b, a\beta + b$.

(ii) If $x^2 + px + q = 0$ and $x^2 + mx + n = 0$ have a root in common, show that this root is the square root of $(pn - qm)/(m - p)$. (Oxf. Sch. Certif.)

20. Observations are taken of the rate of flow of water in a stream, as follows :

October	-	-	-	-	1	3	6	11	14
Thousands of gallons per sec.					6.0	7.4	7.8	3.4	3.0

Plot a graph of these rates of flow. Estimate from your graph the greatest rate of flow, and the mean rate of flow during the first 14 days of October. (Oxf. and Camb. Sch. Certif.)

21. Find for what values of a the coefficient of x^4 in the expansion of $(1 + x + ax^2)^{10}$ is 1695.

For what range of values of a is the coefficient of x^4 smaller than 1695 ? (Lond. Matric.)

22. (a) Resolve into four factors $x^6 - 7x^3 - 8$.

(b) The factors of $12x^2 + 7xy - 12y^2 + 50y - 48$ are

$$(3x + 4y + a)(4x - 3y + b),$$

where a and b are numbers, positive or negative.

Find a and b .

(Scot. Leaving.)

23. Prove that $\log_a x^n = n \log_a x$.

If $y^2 = ax^7$ and $y^7 z^3 = a^3$, express x in its simplest form in terms of a and z . Find y and z when $a = 2.475$, $x = 0.72$, as accurately as the tables permit. (Joint Matric. Bd.)

24. The electrical resistance of a copper wire of circular cross-section varies as its length and inversely as the square of the thickness. Two copper wires have equal resistances and the length of one is twice that of the other. Show that the one is four times as heavy as the other.

(Oxf. and Camb. Sch. Certif.)

25. Each term of the series 109, 2, 91, 33, etc., is formed by adding together corresponding terms of an Arithmetical and a Geometrical Progression, and the first term of the G.P. exceeds the first term of the A.P. by 19. Find the A.P. and the G.P., and the sixth term of the series.

(Oxf. Sch. Certif.)

26. If $y = (x^2 - 2)(x^2 - 3)$,

(i) Calculate the values of x for which y is a maximum or minimum, and the corresponding values of y .

(ii) Prove that y is positive when $x^2 < 2$ or when $x^2 > 3$, and that y is negative when $2 < x^2 < 3$.

(iii) Prove that the graph of y is symmetrical with regard to the axis $x = 0$.

(iv) Sketch the graph. (Oxf. and Camb. Sch. Certif.)

27. Expand by the Binomial Theorem $(x - 2a)^6$.

In the expansion of $(1 + x)^{35}$ show that the coefficient of the 13th term is double the coefficient of the 12th term.

Use the Binomial Theorem to find the coefficient of x^3 in the expansion of $(1 + 5x + 2x^2)^8$.
(Joint Matric. Bd.)

28. Express as the sum of three partial fractions :

$$\frac{(q-r)(r-p)(p-q)}{(x-p)(x-q)(x-r)}$$

and, as the sum of two partial fractions with real quadratic denominators :

$$\frac{2x^2}{x^4 + x^2 + 1}.$$

(Scot. Cert.)

29. Solve the simultaneous equations

$$\frac{2}{x} + \frac{3}{y} + 1 = 0, \quad 12xy + 1 = 0.$$

[The values of x and y are simple fractions ; do not use decimals.]
(Oxf. and Camb. Sch. Certif.)

30. (i) Divide $x^{1\frac{1}{3}} + x^{-1\frac{1}{3}} - 2$ by $x^{\frac{2}{3}} - x^{-\frac{2}{3}}$.

(ii) If $x = 0.125y^{\frac{3}{2}}z^{\frac{1}{4}}$, express y in terms of x and z .
(Oxf. Sch. Certif.)

31. (a) Explain why the decimal parts of the logarithms of 0.58 and 58 to base 10 are the same.

(b) Evaluate
$$\sqrt[3]{\frac{\frac{1}{4}E^2 \cdot L^{\frac{1}{3}}}{W}}$$

when $E = 32.46, \quad L = 5.86, \quad W = 0.826.$

(c) Find what integral power of 3 is nearest to the tenth power of $\frac{7}{4}$.
(Joint Matric. Board.)

32. Factorize : (i) $a^2 - 9b^2 + 16c^2 - 8ac$; (ii) $a^4 + 16b^4 + 4a^2b^2$.
If $a + b + c = 0$, prove that

$$2(a^4 + b^4 + c^4) = (a^2 + b^2 + c^2)^2.$$

33. What is meant by saying that four quantities are in proportion ?

If four quantities a, b, c, d are in proportion, prove that the ratio of a to d is equal to the ratio of bc to d^2 .

If a, b, c and b, c, d are each in continued proportion, prove that

$$a + b + c + d = \frac{a^4 - b^4}{a^2(a - b)}.$$

(Scot. Cert.)

34. The readings of a speedometer on a motor car at intervals of 10 minutes were 33, 30, 38, 6, 28, 6, 37 miles per hour. Plot these on graph paper, and from your graph estimate the distance travelled in the hour.

Give a commonsense reason why the result cannot be expected to be very accurate.
(Oxf. and Camb. Sch. Certif.)

35. If α and β are the roots of the equation

$$ax^2 + bx + c = 0$$

express $\alpha + \beta$, $\alpha\beta$ and $\alpha^{-3} + \beta^{-3}$ in terms of a, b, c .

Solve the equation :

$$\frac{p-a}{x-a} = 1 + \frac{x-p}{p(x-b)}.$$

(London Matric.)

ANSWERS

XXXIX. a. Page 459.

- | | | |
|------------------------|--------------------|---------|
| 1. 12. | 2. 18. | 3. 120. |
| 4. (i) 144 ; (ii) 132. | 5. 720. | 6. 840. |
| 7. 24, 18. | 8. 120, 24, 96, 6. | 9. 72. |

XXXIX. b. Page 461.

- | | | | | |
|---|------------------------|------------------------|-------------|----------|
| 1. 5040, 120, 2520, 40320, 3024, 60480. | 2. (i) 720 ; (ii) 720. | 3. 6. | 4. 120, 24. | 5. 60. |
| 6. (i) 120 ; (ii) 600. | 7. 144. | 8. 696. | 9. 144. | 10. 480. |
| 12. 6720. | 13. 10368000. | 14. (i) 144 ; (ii) 72. | | |

XXXIX. c. Page 464.

- | | | | | |
|---|------------|--------------|------------|---------------------|
| 1. (i) 1260 ; (ii) 3360 ; (iii) 166320 ; (iv) 64864800, 180, 83160. | 2. 12, 18. | 3. 420, 360. | 4. 302400. | 5. (i) 40 ; (ii) 8. |
| 6. 420. | 7. 120. | 8. 2520. | 9. 48. | 10. 243, 48. |
| 11. 1296. | 12. 81. | 13. 624. | 14. 30. | 15. 64. |

XXXIX. d. Page 470.

- | | |
|---|--------------------------------------|
| 1. 20, 84, 55, 1225, 105. | 2. 1140. |
| 3. 12250. | 4. (i) 210 ; (ii) 63 ; (iii) 35. |
| 5. (i) 126 ; (ii) 63 ; (iii) 203. | 6. 8. |
| 7. 13. | 8. 55200. |
| 9. 327600. | 10. 24000. |
| 11. 1485. | 12. 1000. |
| 13. 35, 201600. | 14. 255. |
| 15. 246. | 16. 1023. |
| 17. 511. | 18. $\frac{30}{10 \mid 12 \mid 8}$. |
| 19. $\frac{52}{(13)^4 \mid 4}$; (ii) $\frac{52}{(13)^4}$. | 20. 480. |
| 21. 80. | 22. 2903040. |
| 23. 2880. | 24. $\frac{mn}{(m)^n}$. |
| 25. 315. | 26. (i) 40 ; (ii) 116. |
| 27. 5760. | 28. $(n-2) \mid n-1$. |
| 29. (i) 60 ; (ii) 15120. | 30. (i) 378 ; (ii) 120. |
| 31. $(n+1)^m - 1$. | 32. (i) 15 ; (ii) 208. |
| 33. 70, 35. | 34. $(n-2)(n-3) \mid n-2$. |
| 35. 18480. | 36. (i) 113 ; (ii) 2190. |
| 37. 1343. | 38. |

XLI. a. Page 482.

- | | |
|---|--|
| 1. (i) $x^3 + 5x^2 + 2x - 8$; | |
| (ii) $x^4 + 6x^3 - 21x^2 - 74x + 168$; | (iii) $x^4 - 41x^2 + 400$; |
| (iv) $a^3 + 6a^2b - 37ab^2 - 90b^3$. | |
| 2. $x^4 + 8x^3 + 24x^2 + 32x + 16$. | 3. $x^5 + 15x^4 + 90x^3 + 270x^2 + 405x + 243$. |
| 4. $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$. | |
| 5. $a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + 35a^3x^4 + 21a^2x^5 + 7ax^6 + x^7$. | |
| 6. $1 + 8b + 28b^2 + 56b^3 + 70b^4 + 56b^5 + 28b^6 + 8b^7 + b^8$. | |
| 7. $1 - 10y + 40y^2 - 80y^3 + 80y^4 - 32y^5$. | |
| 8. $1 + 3x + \frac{15}{4}x^2 + \frac{5}{2}x^3 + \frac{15}{16}x^4 + \frac{3}{16}x^5 + \frac{1}{64}x$. | |

9. $16x^4 + 16x^3y + 6x^2y^2 + xy^3 + \frac{y^4}{16}$.
10. $64 - 96x + 60x^2 - 20x^3 + \frac{15x^4}{4} - \frac{3x^5}{8} + \frac{x^6}{64}$.
11. $a^7 - \frac{21a^6}{b} + \frac{189a^5}{b^2} - \frac{945a^4}{b^3} + \frac{2835a^3}{b^4} - \frac{5103a^2}{b^5} + \frac{5103a}{b^6} - \frac{2187}{b^7}$.
12. $x^5 - \frac{5x^3}{2} + \frac{5x}{2} - \frac{5}{4x} + \frac{5}{16x^3} - \frac{1}{32x^5}$.
13. $\alpha^9x^9 + 9\alpha^7x^8y + 36\alpha^5x^7y^2 + 84\alpha^3x^6y^3 + 126\alpha x^5y^4$
 $+ \frac{126x^4y^5}{\alpha} + \frac{84x^3y^6}{\alpha^3} + \frac{36x^2y^7}{\alpha^5} + \frac{9xy^8}{\alpha^7} + \frac{y^9}{\alpha^9}$
14. $2(5a^4b + 10a^2b^3 + b^5)$.
15. $2(729 + 4860x^2 + 2160x^4 + 64x^6)$.
16. $2(x^4 + 18x^2 + 9)$.
17. $140\sqrt{2}$.
19. $x^6 - 3x^5 - 3x^4 + 11x^3 + 6x^2 - 12x - 8$.
20. $1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8$.
21. $1 - 6a + 21a^2 - 44a^3 + 63a^4 - 54a^5 + 27a^6$.
22. (i) $a^8 + 4a^7x + 4a^6x^2 - 4a^5x^3 - 10a^4x^4 - 4a^3x^5 + 4a^2x^6 + 4ax^7 + x^8$;
(ii) $1 + x - 8x^2 - 2x^3 + 25x^4 - 11x^5 - 26x^6 + 28x^7 - 8x^8$.
23. $280x^3$.
24. $-448y^5$.
25. $43750a^4b^4$.
26. $5440x^3$.
27. $\frac{210}{x^6}$.
28. $\frac{5103x^4a^5}{16}$.
29. (i) $20x^3$; (ii) $\frac{2835}{8}$.
30. $126x, -\frac{126}{x}$.
31. -252 .
32. $\frac{2300b^{22}}{x^{16}}$.
33. 3360 .
34. $-\frac{1001a^9}{256}$.
35. $70, -56$.
36. 7920 .

XLI. b. Page 486.

1. The 20th.
2. The 5th.
3. The 12th and 13th.
4. The 6th.
5. The 5th.
6. The 4th and 5th, $=\frac{2}{9}\frac{2}{4}$.
7. The 4th = 6048.
9. (i) $\frac{2}{2}\frac{8}{7}\frac{1}{6}$; (ii) 4725000.
10. 5.
11. 22.
12. 4096.
13. 3125.
22. -285.
23. $729 - 2916x + 4860x^2 - 4320x^3 + 2160x^4 - 576x^5 + 64x^6$.
24. $-167960x^7$.
25. $1 + 1 + \frac{x-1}{2x} + \frac{(x-1)(x-2)}{6x^2} + \dots$; $(r+1)^{\text{th}}$ term = $\frac{\lfloor x \rfloor}{r \lfloor x-r \rfloor} \cdot \frac{1}{x^r}$; 2.495.
26. $\frac{\lfloor n \rfloor}{r-1 \lfloor n-r+1 \rfloor} \alpha^{n-r+1} (2x)^{r-1}, \frac{\lfloor n \rfloor}{r-1 \lfloor n-r+1 \rfloor} \alpha^{r-1} (2x)^{n-r+1}$.
27. -105.
29. 1.061.
30. 49.
31. 56.
32. $840a^4$.
35. $\frac{\lfloor n \rfloor}{\frac{1}{5}(3n+r) \lfloor \frac{1}{5}(2n-r) \rfloor}$.

XLI. c. Page 492.

2. $1 + \frac{3}{4}x - \frac{3}{32}x^2 + \frac{5}{128}x^3$.
3. $1 - \frac{2}{5}x - \frac{3}{25}x^2 - \frac{8}{125}x^3$.
4. $1 - 6x + 27x^2 - 108x^3$.
5. $1 - 3x^2 + 6x^4 - 10x^6$.

6. $1 - 12x + 90x^2 - 540x^3$. 7. $\frac{1}{8} - \frac{3}{16}x + \frac{3}{16}x^2 - \frac{5}{32}x^3$.
8. $1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3$. 9. $a^{-\frac{3}{2}}\left(1 + \frac{3x}{a} + \frac{15x^2}{2a^2} + \frac{35x^3}{2a^3}\right)$.
10. $1 - \frac{5}{2}x + \frac{15}{8}x^2 - \frac{5}{16}x^3$. 11. $3\left(1 + \frac{1}{9}x - \frac{1}{16}x^2 + \frac{1}{1458}x^3\right)$.
12. $4\left(1 + a - \frac{1}{4}a^2 + \frac{1}{6}a^3\right)$. 13. $\frac{3}{1}\frac{1}{2}\frac{5}{8}x^4, -\frac{2}{6}\frac{3}{5}\frac{0}{5}\frac{4}{3}\frac{5}{6}x^9$.
14. $\frac{9}{2}x^2, \frac{7}{2}\frac{7}{56}x^{10}$. 15. $-4x^3, (-1)^r(r+1)x^r$.
16. $-\frac{2}{10}\frac{1}{24}x^6, -\frac{1 \cdot 3 \cdot 5 \dots (2r-3)}{2^r \lfloor r \rfloor}x^r$.
17. $\frac{b^r}{a^{r+1}}x^r, -\frac{(n-1)(2n-1) \dots \{(r-1)n-1\}}{\lfloor r \rfloor}x^r$.
18. $(-1)^{r-3}\frac{5 \cdot 3 \cdot 1 \cdot 1 \cdot 3 \dots (2r-7)}{2^r \lfloor r \rfloor}x^r, (-1)^r\frac{3 \cdot 5 \cdot 7 \dots (2r+1)}{\lfloor r \rfloor}x^r$.
19. $\frac{(r+1)(r+2)(r+3)(r+4)}{\lfloor 4 \rfloor}x^r$. 20. $(-1)^r\frac{(r+1)(r+2)}{2}x^{2r}$.
21. $(-1)^r\frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2r-1)}{2^r \lfloor r \rfloor}x^r$. 22. $(-1)^r\frac{2 \cdot 5 \cdot 8 \dots (3r-1)}{3^r \lfloor r \rfloor}x^r$.
23. $\frac{(r+1)(r+2) \dots (r+9)}{\lfloor 9 \rfloor} \frac{x^r}{2^{r+10}}$.
24. $(-1)^r\frac{p(p+q)(p+2q) \dots (p+r-1)q}{\lfloor r \rfloor} \left(\frac{x}{q}\right)^r$.
25. $(-1)^r\frac{1 \cdot 3 \cdot 5 \dots (2r-1)}{\lfloor r \rfloor}x^r$. 26. $\frac{2 \cdot 5 \cdot 8 \dots (3r-1)}{\lfloor r \rfloor}x^r$.
27. $-\frac{2 \cdot 1 \cdot 4 \dots (3r-5)}{3^r \lfloor r \rfloor}x^r$. 28. (i) $-\frac{5}{8}x^4$; (ii) $-\frac{4}{81}x^3$; (iii) $-\frac{2}{3}x^5$.

XLI. d. Page 493 1. The 7th. 2. The 2nd. 3. The 38th and 39th.

4. The 1st and 2nd. 5. The 3rd. 6. The 4th. 7. (i) $\frac{5}{4}$; (ii) $\frac{7}{2} \cdot 3^{\frac{3}{4}}$.
8. (i) 4; (ii) 201; (iii) -5050.
9. (i) $3r$; (ii) $\frac{1}{2}(r^2+r+2)$; (iii) $(-1)^{r-1}(2r-1)$.
11. $(-1)^r\frac{(r+1)(r+2)(5r+6)}{6}$. 12. $\frac{1}{2}\sqrt{3}$. 13. $\sqrt[3]{2}$.
18. $\frac{3}{2}$; the series is the expansion of $(1 - \frac{1}{3})^{-5} \times (\frac{2}{3})^4$.
22. (i) $\frac{1 \cdot 3 \cdot 5 \dots (2r-1)}{2^r \lfloor r \rfloor}$; (ii) $\frac{(r+1)(r+2)(r+3)}{\lfloor 3 \rfloor}$; (iii) $(-1)^r$.

(iv) $\frac{\lfloor 2n \rfloor}{\lfloor r \rfloor \lfloor 2n-r \rfloor}$.

- XLI. e. Page 498.** 1. 1.08243. 2. 0.85873. 3. 7.07106.
4. 10.0100. 5. 6.0092. 6. 1.9743. 7. 4.9900.
8. 9.9933. 9. 0.1459. 10. 0.1995. 11. 125.1500.
12. 1.0013. 13. 1.414214. 14. $1 + \frac{1}{6}x$. 15. $4 + \frac{2}{48}x$.

16. $1 - \frac{5}{6}x$. 17. $1 - \frac{7}{60}x$. 18. $1 - \frac{5}{4}x$. 19. $1 + 2x$.
 20. $16(1 - \frac{41}{40}x)$. 21. $1 - \frac{35}{192}x$. 22. $1 - 3x + \frac{35}{4}x^2$.
 23. 1.0032. 24. 0.9974. 25. 1.0054. 26. 1.00017. 27. 0.99284.
 28. 1.00048. 29. 1.0076. 30. 0.676. 31. 1.0002. 32. 1.017.
 33. 1.06. 34. 1.0015. 35. 0.9988. 36. 4.5%. 37. 0.5 sq. cm.
 38. 1003 sq. ft. 40. 0.0000009. 41. 360. 42. 1.3 in excess.
 43. 2.6% in defect. 44. 7.8% in excess. 45. $\frac{5.11.13x^9}{2^{16}}$.
 46. $\frac{1}{6}n(n-1)(n-38)$. 47. $\frac{1}{\sqrt{2}}\left(\frac{1}{4} + \frac{11x}{16} + \frac{179}{128}x^2\right)$.
 48. (i) -73; (ii) $-(8n^2 + 16n + 9)$. 49. $\frac{1.3.5 \dots (2n-1)}{n} 2^n = \frac{\lfloor 2n \rfloor}{\lfloor n \rfloor \lfloor n \rfloor}$.
 51. $\frac{n}{2a^2}\{2a - (n+1)b\}$. 52. $\sqrt{10} = 3.162$; $\sqrt{26} = 5.099$; $\sqrt{123} = 11.091$.

XLII. Page 504.

1. $\frac{3}{x+5} + \frac{2}{x-3}$. 2. $\frac{15}{x-1} - \frac{11}{x-2}$.
 3. $\frac{8}{x+4} + \frac{6}{x-3}$. 4. $\frac{3}{1-2x} + \frac{5}{1+3x}$. 5. $\frac{3}{x-1} - \frac{3}{x-2} + \frac{1}{x+3}$.
 6. $\frac{3}{2-x} + \frac{2x-1}{x^2+x+1}$. 7. $\frac{1}{x} + \frac{2}{x+1} - \frac{3}{(x+1)^2}$.
 8. $\frac{2}{x+2} - \frac{1}{x-1} + \frac{3}{(x-1)^2}$. 9. $x+2 + \frac{2}{x-3} + \frac{3}{x+1}$.
 10. $\frac{3}{x+2} - \frac{2}{x+3} - \frac{4}{(x+3)^2}$. 11. $-\frac{1}{3(1+2x)} + \frac{5}{3(x+2)} - \frac{4}{(x+2)^2}$.
 12. $\frac{15x+2}{x^2+1} - \frac{12}{x+6}$. 13. $\frac{1}{1-x} + \frac{2}{1-2x}$; $(1+2^{r+1})x^r$.
 14. $\frac{1}{x-5} - \frac{1}{x-3}$; $\left(\frac{1}{3^{r+1}} - \frac{1}{5^{r+1}}\right)x^r$.
 15. $\frac{1}{2+x} + \frac{3}{1+3x}$; $(-1)^r \left\{ \frac{1}{2^{r+1}} + 3^{r+1} \right\} x^r$.
 16. $\frac{3}{1+2x} - \frac{5}{1-x}$; $\{(-1)^r 3 \cdot 2^r - 5\} x^r$.
 17. $\frac{1}{1-x} - \frac{1}{1+x} - \frac{4}{1-2x}$; $\{1 + (-1)^{r-1} - 2^{r+2}\} x^r$.
 18. $\frac{1}{2(1-x)} + \frac{1}{6(1+x)} - \frac{4}{3(2-x)}$; $\left\{ \frac{1}{2} - \frac{1}{3 \cdot 2^{r-1}} + \frac{(-1)^r}{6} \right\} x^r$.
 19. $\frac{1}{4(1+4x)} + \frac{11}{4(1+4x)^2}$; $(-1)^r (12 + 11r) 4^{r-1} x^r$.
 20. $\frac{2}{1+x} + \frac{3}{(1+x)^2} - \frac{6}{2+3x}$; $(-1)^r \left(3r + 5 - \frac{3^{r+1}}{2^r} \right) x^r$.
 21. $\frac{4}{2-x} - \frac{9}{4(3-x)} - \frac{7}{4(1-x)} + \frac{1}{2(1-x)^2}$; $\left(\frac{2r-5}{4} + \frac{1}{2^{r-1}} - \frac{1}{4} \cdot \frac{1}{3^{r-1}} \right) x^r$.

$$22. \frac{1}{x-4} + \frac{2}{(x-4)^2} - \frac{3}{(x-4)^3}; \frac{1}{x+2} - \frac{3}{(x+2)^2} - \frac{2}{(x+2)^3} + \frac{4}{(x+2)^4}.$$

$$23. \frac{1}{3} \left(\frac{1}{3n-2} - \frac{1}{3n+1} \right); \frac{n}{3n+1}.$$

Miscellaneous Examples IX. Page 505. 1. 30.

2. $\frac{x-16}{19x+18}, \frac{18y+16}{1-19y}$. 3. 174; $15x-4, 8x+7$.
4. $167, \frac{1}{8}\frac{4}{3}$. 5. -5103. 6. $x=1, y=2; x=\frac{4}{3}, y=\frac{5}{8}$.
7. $y=\frac{4}{3}$, when $x=\frac{2}{3}$. $x=-\frac{1}{3}, y=\frac{1}{3}$.
8. (i) $(mt+n)(m-nt)$; (ii) $(x^2+3xy+y^2)(x^2-3xy+y^2)$.
9. (i) $\pounds \frac{c-a+b}{2}$ in first, $\pounds \frac{c+a-b}{2}$ in second;
 (ii) $\pounds \frac{2c-a+2b}{3}$ „ $\pounds \frac{c+a-2b}{3}$ „
10. (i) $\frac{1}{2}$; (ii) $a=1.25, c=1$. 11. 7.8125 gallons. 13. 9.
14. $m+1$. 16. $\frac{2880q-23bp}{460(12q-p)}$ shillings. 17. 1.24 secs.
18. $2x^2-9x-12=0$. 19. $9r+5$. 21. $A=2, B=3, C=-4$.
22. $\pounds\{(a-b)x+a\}$. 23. (i) 0, $-\frac{1}{2}\frac{7}{8}$; (ii) $\frac{q^2}{p}, -\frac{p^2}{q}$.
24. $64\{(-\frac{1}{2})^n-1\}; 16$. 27. 1, 1.414, 2, 2.828, 4, 5.657, 8.
29. (i) $(\frac{1}{2})^{2n-2}$; (ii) $\sqrt{6}-\sqrt{2}$. 33. 0.002 too small. (i) 1.809; (ii) 0.691.
34. 1 mile; $A, 4' 57''; B, 5' 30''$. 36. $\frac{a^2+ab+b^2}{a+b}$. 37. 0, $\frac{20}{3}$.
38. $7+4\sqrt{3}; 2+\sqrt{3}$. 40. $n^2(b-a)$. 41. 7 or 14.
42. -0.97. 43. $x=\frac{b+c}{2a}, y=\frac{c+a}{2b}, z=\frac{a+b}{2c}$. 44. 1, 2, 4, 8, 16, ...
45. $\frac{ac-b^2}{a+c-2b}$. 47. 42. 48. About 17.4 yrs. 49. $\frac{1}{2}\sqrt{6}$.

XLIII. Page 516. 1. (i) $\frac{(-1)^r}{|r|}(1+r)$; (ii) $\frac{(-1)^{r-1}}{|r|}(ar-b)$. 2. $\frac{e^a b^n}{|n|}$.

4. $\frac{(-1)^r}{|r|}(1+2r-r^2)$. 8. 1. 9. $e-1$. 10. $\frac{3}{2}e$. 11. $e^{a^2}-e^{b^2}$.

12. $\frac{1}{2}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\right)$. 14. $-y+\frac{y^2}{2}-\frac{y^3}{3}+\dots$. 15. 0.69315.

16. $x-\frac{5x^2}{2}+\frac{7x^3}{3}-\frac{17x^4}{4}; \frac{(-1)^{r-1}2^r-1}{r}x^r$.

17. $\frac{2^{2r}(-1)^{r-1}-2^r}{r}x^r; 2x-10x^2+\frac{56x^3}{3}-68x^4$.

18. $\frac{(-1)^{r-1}+3^r}{r}x^r$. 20. $3x-\frac{5x^2}{2}+3x^3-\frac{17x^4}{4}+\dots$.

21. 0.0020000007 .

22. $\frac{x}{1-x} + \log_e(1-x)$.

23. Each series may be shewn to be equal to $\frac{1}{2}(1 - 3 \log 2)$.24. $\log 13 = 1.11394$, $\log 17 = 1.23045$. [For $\log 13$, put $n=12$ in the series of Art. 552. For $\log 17$, put $n=50$ in the series (1) of Art. 554. This gives $\log 51$, or $\log 3 + \log 17$.]

25. 5569.

XLIV. Page 522. 1. £985. 2. £737. 3. £1793. 4. 86 yrs.
 5. £2184. 6. 9.6 nearly. 7. £9868. 9. £3755. 12s. 6d.
 10. £11708. 8s. 11. £588. 12. £1548. 9s. 13. £2231. 12s. 4d.
 14. $4\frac{1}{2}\%$. 15. 16. 16. £1020. 17. £1340. 1s. 11d. 18. £4200.

XLV. Page 530. 1. 22220. 2. 1380755. 3. 28091. 4. 251021.
 5. (i) 244332343; (ii) 14320241. 6. 564, 26. 7. 345, 3024.
 8. 32552, 3552, 215312024. 9. 1342, 34705. 10. 30523.
 11. 123456. 12. $12^3 + 12^2 + 12$. 13. $\cdot 27$. 14. (i) $\cdot 0203$; (ii) $\cdot 106314$.
 15. $132 \cdot 12$. 16. $\cdot 26$. 17. $200 \cdot 211$. 18. (i) $\frac{21}{50}$; (ii) $\frac{111}{302}$.
 19. 1892, 292. 20. Five, Nine. 21. Nine. 22. Twelve.
 23. 7.e. 24. 1 lb., 2 lbs., 32 lbs., 64 lbs., 128 lbs. 25. $\frac{58}{105}$.

XLVI. Page 533. 9. All except such as lie between -2 and $+1$.
 13. Unless $a=b=c$, or $a+b+c=0$.

XLVII. a. Page 537. 1. $\frac{ab}{b-a}$. 2. $\frac{cd(a+b) - ab(c+d)}{ab - cd}$.
 3. $\frac{a+b+3}{2}$. 4. $ab - bc - ca$. 5. $2ab - bc - 2ca$. 6. $\frac{ab + bc + ca}{abc}$.
 7. $\frac{pq}{p-q}$. 8. $\frac{ab + c^2}{a - b - 2c}$. 9. $-2(a+b+c)$. 10. 0, $\frac{a^2 + b^2 + a^3 + b^3}{a + b + a^2 + b^2}$.
 11. 2, -1 . 12. 0, 3. 13. $9, \frac{1}{9}$. 14. $\frac{4}{9}, \frac{1}{4}$. 15. 0, -3 .
 16. ± 1 . 17. 1, 2, $\frac{1}{2}$, 4. 18. 3, $-1, \frac{3 \pm \sqrt{21}}{2}$. 19. 16, $-\frac{4}{11}$.
 20. $\frac{a(b^2 - c^2)}{b^2 + c^2}$. 21. 25, -3 . 22. 20, 1. 23. 2, $\frac{1}{2}$. 24. $-2a$.
 25. 3, $-\frac{1}{2}$. 26. 5, $\frac{1}{2}$. 27. 2, $-\frac{14}{3}$. 28. 3, $\frac{5}{87}$. 29. $\frac{21}{2}$.
 30. 7, $-\frac{31}{3}$. 31. 2, $-\frac{1}{3}$. 32. 2, $\frac{1}{2}$, 3, $\frac{1}{3}$. 33. 2, $\frac{1}{2}$, $2 \pm \sqrt{3}$.
 34. 2, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{2}{3}$. 35. $-2, \frac{1}{2}, -3, \frac{1}{3}$. 36. 2, $-4, -1 \pm \sqrt{71}$.
 37. $-a, 2a, -5a, 6a$. 38. 0, $\pm \frac{3}{10}$. 39. $\frac{9}{2}$, 10, -1 .
 40. $\pm 4\sqrt{2}$. 41. 0, $\frac{63a}{65}$. 42. 1, $\frac{(\sqrt{a} - \sqrt{b})^2 + 4}{(\sqrt{a} + \sqrt{b})^2 - 4}$.

XLVII. b. Page 542.

1. $x=3, 4, \frac{1}{2}(7 \pm \sqrt{-295})$;
 $y=4, 3, \frac{1}{2}(7 \mp \sqrt{-295})$.
2. $x=4, -2, 1 \pm \sqrt{-15}$;
 $y=2, -4, -1 \pm \sqrt{-15}$.
3. $x=4, -1, \frac{1}{2}(3 \pm \sqrt{-43})$;
 $y=-1, 4, \frac{1}{2}(3 \mp \sqrt{-43})$.
4. $x=0, \frac{10}{3}$;
 $y=\frac{19}{7}, \frac{1}{3}$.
5. $x=5, -5, \pm \frac{2}{5}\sqrt{145}$;
 $y=-2, 2, \mp \frac{1}{5}\sqrt{145}$.
6. $x=2, y=-1$; $x=-\frac{6}{5}, y=\frac{3}{5}$.
7. $x=2, y=1$; $x=-\frac{17}{5}, y=-\frac{13}{5}$.
8. $x=7, y=3$; $x=-\frac{35}{4}, y=-\frac{15}{4}$.
9. $x=y=\frac{3}{4}$; $x=0, y=1$; $x=1, y=0$.
10. (i) $x=3, 2, -3, -2$; $y=-2, -3, 2, 3$.
(ii) $x=3, y=1$; $x=-1, y=-3$.
11. $x=5, y=1$; $x=\frac{21}{5}, y=\frac{7}{5}$.
12. $x=2, \frac{1}{2}, x=-2, -\frac{1}{2}$,
 $y=5$; $y=-5$.
13. $x=8, y=2$; $x=3, y=7$.
14. $x=\pm 1, y=\pm 6, z=\pm 3$.
15. $x=\pm 3, y=\pm 4, z=\pm 6$.
16. $x=\pm 12, y=\pm 6, z=\pm 8$.
17. $x=10, y=6, z=4$; $x=-\frac{10}{21}, y=-\frac{2}{7}, z=-\frac{4}{21}$.
18. $x=3, y=2, z=1$; $x=-1, y=-12, z=-17$.
19. $x=4, y=5, z=3$; $x=-6, y=-7, z=-5$.
20. $x=\pm 2, y=\pm 1, z=\pm 3$.
21. $x=\pm\sqrt{2}, y=\pm\sqrt{3}, z=\pm\sqrt{6}$.
22. $x=12, y=6, z=3$; $x=3, y=6, z=12$.
23. $x=6, y=9, z=4$; $x=6, y=4, z=9$.
24. $x=-5, y=3, z=1$; $x=-5, y=1, z=3$.
25. $x=\pm \frac{a(b^2+c^2)}{2bc}$, $y=\pm \frac{b(c^2+a^2)}{2ca}$, $z=\pm \frac{c(a^2+b^2)}{2ab}$.
26. $x=\pm 4, y=\pm 8, z=\mp 6$.
27. $x=3, y=2, z=1$; $x=1, y=2, z=3$;
 $x=\frac{-1 \pm \sqrt{29}}{2}$, $y=-3$, $z=\frac{-1 \mp \sqrt{29}}{2}$.

XLVII. c. Page 546.

1. $x=7, 4, 1$; $y=2, 7, 12$.
2. $x=11, 7, 3$; $y=2, 9, 16$.
3. $x=3, 8, 13$; $y=6, 4, 2$.
4. $x=2, y=4$.
5. $x=10, y=8$.
6. $x=26, 13$; $y=8, 19$.
7. $x=3, y=2$.
8. $x=17, 12, 7, 2$; $y=1, 4, 7, 10$.
9. $x=12, 1$; $y=2, 6$.
10. $x=13p-3, 10$;
 $y=6p-2, 4$.
11. $x=5p-2, 3$;
 $y=7p+3, 10$.
12. $x=21p+10, 10$;
 $y=8p+2, 2$.
13. $x=7p+3, 10$;
 $y=8p-1, 7$.
14. $x=11p-3, 8$;
 $y=7p-5, 2$.
15. $x=13p+15, 15$;
 $y=10p+8, 8$.
16. Of the first 28, 16, or 4; of the second 2, 9, or 16.
17. 13.
18. 140, 12; 56, 96.
19. $\frac{5}{7}, \frac{8}{11}$.
20. 8 florins, 1 half-crown. An infinite number.

21. 30 tables, 9 sofas ; or 5 tables, 32 sofas.
 22. 89, 23 ; 65, 47 ; 41, 71 ; 17, 95. 23. $x=3, y=2, z=5$.
 24. $x=4, 7, 10, \dots$;
 $y=2, 7, 12, \dots$;
 $z=7, 13, 19, \dots$.
 25. 26 rams, 4 pigs, 6 oxen ; or 11 rams, 17 pigs, 8 oxen.

Miscellaneous Examples X. Page 547.

1. $(2a-3b)(a+b)$.
 2. $(2a+3b)(2a-3b)(x-2a)(x^2+2ax+4a^2)$. 4. 30240.
 5. (i) $2, 3, \frac{5 \pm \sqrt{37}}{2}$; (ii) $20\frac{3}{4}, 16\frac{3}{4}$. 7. $\log 2$. 8. 2449440.
 9. (i) $(x^2+2x+3)(x^2-2x+3)$; (ii) $(a+b)\{3(a+b)+2\}\{3(a+b)-2\}$.
 10. y^3+3y . 11. (i) $-\frac{a+b}{4}$; (ii) $x=\frac{c}{c+1}, y=-\frac{c}{c+1}$. 12. 0.02 nearly
 14. 360. 15. 2080. $A, 97; B, 163$. 16. $4r+1$.
 17. $A=\frac{1}{2}(1+\sqrt{5}), B=\frac{1}{2}(1-\sqrt{5})$. 19. 1, 2. 20. $-1, -\frac{1}{2}$.
 21. 3.742. 22. $x=bc/(a-b)(a-c), y=ca/(b-c)(b-a), z=ab/(c-a)(c-b)$.
 24. £468. 11s. 2d.; £1000. 26. $x=-b, y=a$. 27. (i) 8; (ii) 81.
 28. Cost price 8s.; Sale price 9s. 29. ab sq. ft. 30. 14.
 32. $\bar{3}.698970; 0.799340; \bar{1}.785248$. 33. $p-q-\frac{pq}{100}$. 35. $\frac{7}{4}$.
 37. $3\frac{1}{4}$ mi. per hr.; 2 hrs., 36 min. 38. 14, 22. 39. 180.
 40. $\{2 \cdot 3^r + (-1)^{r+1}\}x^r$. 41. $y=\frac{(b+p)(b+q)}{b-a}$.
 44. (i) $\frac{\sqrt{2}}{2}(7-\sqrt{5})$; (ii) $\sqrt{\frac{a^2+ax+x^2}{2}} + \sqrt{\frac{a^2-ax+x^2}{2}}$.
 45. 2.83, 4.95, 12.16, 5.5. 46. $A, 12$ mi. per hr.; $B, 7$ mi. per hr.
 47. $a=251, b=1; a=127, b=2; a=55, b=5; a=35, b=10$.
 48. 1.105. 49. $(a+b)(a-b)(a^2+b^2)(2a-b)(2a+b)$. 50. $1-x$. 51. 100.
 52. (i) $-\frac{9}{4}$; (ii) -2 . 53. $\frac{2ax}{a^2-b^2}$. This $> \frac{2ax}{a^2}$, or $\frac{2x}{a}$. 55. 14 miles.
 58. 0. 60. $1+2x+3x^2+\dots; (n+1)x^n-nx^{n+1}; \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$.
 61. $\frac{1}{2}$. 62. $-\frac{1}{6}(n^3-15n^2+2n)$. 63. £439. 19s.
 66. $(x^2+7x+15)(x^2+7x+7)$. 69. (i) $x=3, 2, 5, 1$; (ii) $x=\pm 2\sqrt{3}, \pm 3$;
 $y=2, 3, 1, 5$. $y=\pm 3, \pm 2\sqrt{3}$.
 70. $\frac{231x^{12}}{4}$. 71. $\frac{2}{1-x} + \frac{2x+3}{1+x^2}$. 72. $\frac{e^x-e^{-x}}{2}$.
 73. $\frac{1}{m+1} + \frac{1}{n+1} + \frac{1}{p+1} = 1$, or $mnp=m+n+p+2$. 74. $a+b$.
 75. 69.7 in., 30.3 in. 76. $a^2x^2+(2ac-4b^2)x+c^2=0$.
 78. $3 + \frac{12}{x-2} + \frac{1}{(x-2)^2}$. 79. 642.2. 81. $(2x-3y)(x+2y)(3x-y)$.

82. $\frac{p+q}{p-q}$ 83. -2 84. $x=y=\frac{3}{4}$; $x=0, y=1$; $x=1, y=0$.
 85. 8.40 a.m. at C. 86. 31%. 88. $x=\pm 3, y=\mp 2, z=\pm 1$;
 $x=\pm \frac{11}{\sqrt{19}}, y=\pm \frac{1}{\sqrt{19}}, z=\pm \frac{7}{\sqrt{19}}$.
 91. None. 92. (i) 4; (ii) $3xyz$. 93. $\frac{1}{2}$. 94. 26. 95. 19.
 96. $a=0.32, b=8.8$. 99. $2c-a-b$. 100. $a^3-3ab^2+2c^3=0$.
 101. 4.12 p.m. 103. 59. 104. 2.8% too much. 105. $2+4x-9x^2+3x^3$.
 108. 324. 110. 1680. 111. $8x=\sqrt{yz}+2\sqrt[3]{yz}$.
 113. $a=2, b=3, p=10, q=12$; $a=2, b=-3, p=-2, q=-12$.
 115. $2^{n+1}-2+\frac{n(n^2-1)}{3}$. 117. (i) $-\frac{a+2b}{2}$; (ii) $a+b+c$.
 120. 30 years. £410, £320.

Examples from Recent Examination Papers A. Page 557.

1. $(3x-5y)(3x+5y)$; $(x-9)(x+7)$; $(x+y-z)(x-y-z)$; $(a-b)(a-2c)$.
 2. (i) 6; (ii) $\frac{a-b}{p-q+bp-aq}$. 3. $(a+3b)(2a+b)(3a+b)(2a-b)$.
 4. 148. 5, 5, 9. 6. 20. 7. $10\left\{\left(\frac{6}{5}\right)^n-1\right\}$; 30.
 8. (i) (a) $(x-6)(x+2)$; (b) $(x+1)(x+2)(x-3)$; (ii) $\frac{8(x+2)}{(x-4)(x+5)}$.
 9. (a) $\frac{x^2+1}{x}$; (b) $\frac{2(x^2-x+1)}{x^2-1}$; (c) $\frac{1}{x-1}$.
 10. (i) 2.447(1); (ii) 1.196. 11. (i) $\frac{2b}{a} \cdot \frac{1+c}{1-c} \cdot y$; (ii) $\frac{x}{2y} \cdot \frac{1-c}{1+c} \cdot a$.
 12. (i) 2; (ii) 7, 19. 13. (a) $0 < x < .91$; (b) -1.65 or 0.15 .
 14. $7\frac{1}{2}$ hours. 15. (a) $\frac{1}{2}, -\frac{1}{3}$; (b) $\frac{1}{3}$. 16. (b) $(2x+3a)(x-2a)$.
 17. (i) $\frac{a+x}{ax}$; (ii) $\frac{2}{x+2}$. 18. 1.98; .504; .478.
 19. 2 and 1, or 7 and 4. 20. (i) 243; (ii) 4, 12, 36; 4, -16, 64.
 21. 3.19, .31. 22. $(x-3)(x+1)$. 23. $\frac{8}{15}$. 24. (i) 30; (ii) .3962.
 25. £1. 11s. 26. $\frac{3a}{2b}$. 27. £40. 28. (a) $-\frac{3}{5}$ or $\frac{3}{2}$; (b) 17.
 29. (a) $(x+16)(x-3)$; (b) $(8x-5)(x+2)$; (c) $2(x+2y)(x^2-2xy+4y^2)$.
 30. (i) 1; (ii) (a) 16, 4; (b) 8 or 7. 31. (i) -4; (ii) 13.73.
 32. (i) $\frac{4(y+4)}{y+5}$; (ii) $\frac{16-5x}{x-4}$; $x=3$ or 16, $y=-1$ or $-\frac{16}{3}$.
 33. 23 days. 34. 2.54, -.79. 35. (i) $267\frac{3}{4}$; (ii) $\frac{3}{2}, \frac{6}{7}$.
 36. (a) $(x-9y)(4x+3y)$; (b) $(y-1)^2(y+1)^2$; (c) $a(b+c)(c-d)$.
 37. (i) $\frac{1}{7}$; (ii) 21, 6. 38. 1.570, .3384, 15.70, 33.84, .03384.
 39. 20 days. 41. (a) (i) $+\frac{9q^2}{16}$; (ii) $+\frac{9}{4}$; (b) x^2-15 .

42. 1, ± 3 , 9, ± 27 , 81. 43. $6 - 6a^2$.
44. (i) $3ab(a - 2b)(a + 2b)$; (ii) $(x + 1)(x + y - 1)$.
45. (a) 3, 3; (b) $\frac{ab}{b^2 - a^2}$; (c) -3 or $\frac{1}{2}$. 46. $\cdot 3158$; £30 4s.
47. (a) $z = \frac{kx^2}{y}$. 48. From $x = -3$ to $x = -1.5$; -2.86 , $-.14$. 49. 8 inches.
50. (a) $(2x + y)(x - y)$; (b) $x(x + y)(a - 2b)$; (c) $(2p - q - 2x)(2p - q + 2x)$.
51. (b) 2.9121 , $\bar{3}.8644$, 37.94 . 52. (a) 4 or $\frac{13}{8}$; (b) $\frac{5}{3}$, $\frac{5}{6}$.
54. $14''$, $19''$. 55. $x^9 + 2x^8 + 3x^7$, $8x^2 + 9x + 10$. 57. $-\frac{14}{5}$; $\frac{1}{3}$.
58. (a) $(2x - 3)(5x - 3)(x + 2)$; (b) $\frac{(a - 2x)(2x - 3a)}{(2a - x)(3a + 2x)}$.
59. (a) $\text{£}b\left(1 - \frac{c}{20}\right)$; (b) $273\left(\frac{pv}{cd} - 1\right)$.
60. (i) 7 or $\frac{35}{3}$; (ii) $x = 2$ or $\frac{7}{8}$, $y = 1$ or $\frac{7}{4}$. 61. (c) 47.84 .
62. $x = 0$, -1 or 3 . 63. 507.

Examples from Recent Examination Papers B. Page 565.

1. (i) $x = 1$ or $-\frac{7}{3}$, $y = -1$ or 4 ; (ii) $x = \pm \frac{2}{3}$, $y = \pm \frac{5}{3}$, $z = \pm 1$.
2. (b) $a^3 + 1 + \frac{1}{a^3}$; (c) $a^{\frac{2}{3}} - 3 + 2a^{-\frac{1}{2}}$. 3. 26.4 in.
4. $\cdot 6$, $\cdot 3$. 5. 145 ; 15 and 16 . 6. $\frac{9}{5}$.
7. $1 - 30x + 405x^2 - 3240x^3 + 17010x^4$, $\cdot 9704018$.
8. (a) $x = \pm 2$, $\pm \frac{5}{\sqrt{7}}$; $y = \mp 1$, $\mp \frac{3}{\sqrt{7}}$; (b) 1. 9. $\frac{1}{4}$, $-\frac{1}{6}$, $\cdot 625$.
10. -13 , -5 , 26 . 11. (a) $\frac{ar + b}{r + 1}$, $\frac{br + a}{r + 1}$; (b) $n(10 - n)$.
12. 3 , 5 ; $x^3 + 2x^2 + 3x + 4$. 13. $8d.$, $1s.$ $3d.$
14. $a = 1$ or $-\frac{19}{17}$, $b = -1$ or $\frac{15}{17}$. 15. (b) 64 ; (c) $x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.
16. $x = 1$ or $-\frac{1}{4}$; $y = 3$ or $\frac{1}{2}$. 17. (a) $\log_{10} \frac{15}{2a}$, 75 ; (b) $x = 1.17$, $y = -0.414$.
18. 0 , $\cdot 54$ or -6.54 . 19. $x^2 - bx - a^2 = 0$.
20. About 8000 galls. per sec.; 5700 galls. per sec.
21. 3 or -11 ; between 3 and -11 .
22. $(x - 2)(x + 1)(x^2 + 2x + 4)(x^2 - x + 1)$; -6 , 8 .
23. $x = a^{-\frac{1}{49}}z^{-\frac{6}{49}}$; $y = 0.498$, $z = 12.58$.
25. 45 , 50 , 55 , 60 ; 64 , -48 , 36 , -27 ; $54\frac{13}{16}$.

26. (i) $x=0, \pm 1.58$; $y=6, -.25$.

27. $x^6 - 12x^5a + 60x^4a^2 - 160x^3a^3 + 240x^2a^4 - 192xa^5 + 64a^6$; 7560.

28. $\frac{r-q}{x-p} + \frac{p-r}{x-q} + \frac{q-p}{x-r}$; $\frac{x}{x^2-x+1} - \frac{x}{x^2+x+1}$.

29. $x = \frac{1}{4}$ or $-\frac{2}{9}$; $y = -\frac{1}{3}$ or $\frac{3}{8}$.

30. $x^{\frac{2}{3}} - x^{-\frac{2}{3}}$; $4x^{\frac{2}{3}}z^{-\frac{1}{6}}$.

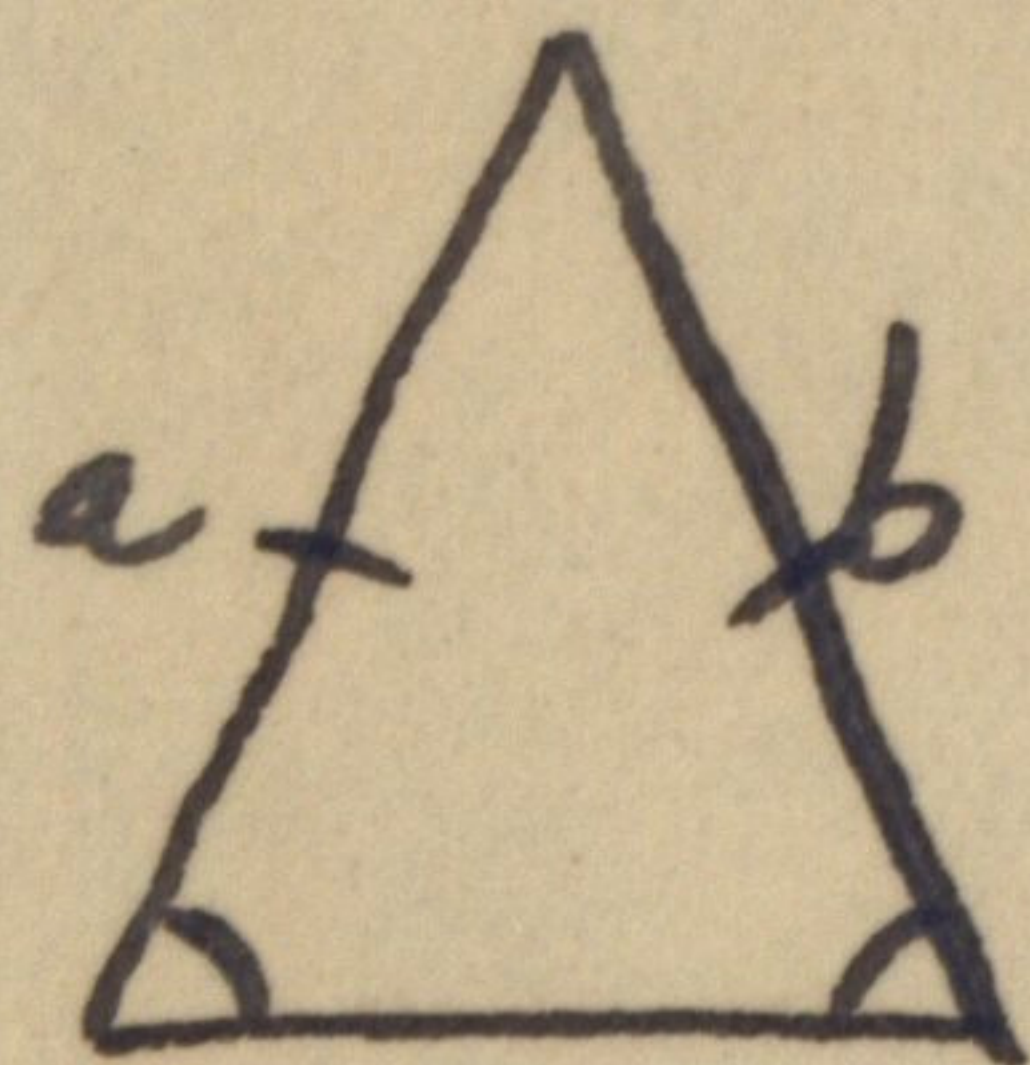
31. (b) 8.3; (c) 5th.

32. (i) $(a-3b-4c)(a+3b-4c)$; (ii) $a^2 - 2ab + 4b^2)(a^2 + 2ab + 4b^2)$.

34. About 24 miles.

35. $-\frac{b}{a}, \frac{c}{a}, -\frac{b}{c^3}(b^2 - 3ac)$; $x=p$ or $\frac{pb+a}{p+1}$.

Bill Nelson



$$a = b$$

$$ab = b^2$$

$$ab - a^2 = b^2 - a^2$$

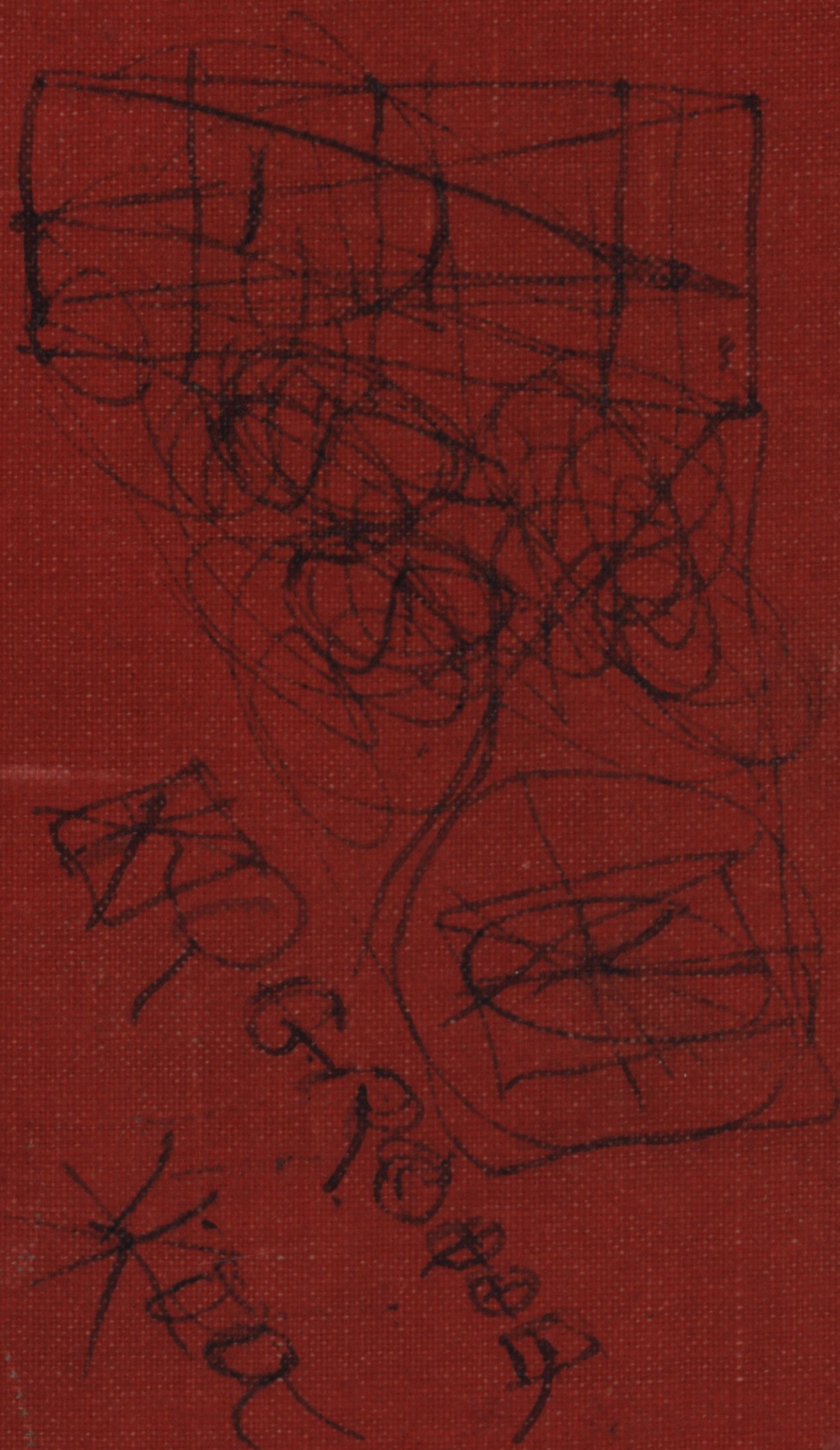
$$a(\cancel{b-a}) = (\cancel{b-a})(b+a)$$

$$a = b + a$$

$$a = a + a$$

$$a = 2a$$

$$1 = 2$$



HIST